3767 *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let R, r be the circumradius and inradius of a right-angled triangle. Prove that

$$\frac{R}{r} + \frac{r}{R} \geq 2\sqrt{2}$$

Solution by Ercole Suppa, Teramo, Italy

Let $b = CA$, $c = AB$ be length of the legs, let $a = BC$ be the length of the hypotenuse, let S, p be the area and semiperimeter of triangle ABC respectively.

Clearly we have $R = a/2$ and the well known formula of inradius yields

$$r = \frac{S}{p} = \frac{bc}{a+b+c} = \frac{bc(b+c-a)}{(a+b+c)(b+c-a)} = \frac{bc(b+c-a)}{(b+c)^2 - a^2}$$
$$= \frac{bc(b+c-a)}{b^2+c^2+2bc-a^2} = \frac{bc(b+c-a)}{2bc} = \frac{b+c-a}{2}$$

Therefore the required inequality is equivalent to

$$\begin{gathered}
\frac{R}{r} + \frac{r}{R} \geq 2\sqrt{2} \quad \Leftrightarrow \\
R^2 + r^2 - 2\sqrt{2}Rr \geq 0 \quad \Leftrightarrow \\
\left(R - \sqrt{2}r\right)^2 \geq r^2 \quad \Leftrightarrow \\
R - \sqrt{2}r \geq r \quad \Leftrightarrow \\
\frac{R}{r} \geq \sqrt{2} + 1 \quad \Leftrightarrow \\
\frac{a}{b+c-a} \geq \sqrt{2} + 1 \quad \Leftrightarrow \\
\left(\sqrt{2} - 1\right) a \geq b + c - a \quad \Leftrightarrow \\
\sqrt{2}a \geq b + c
\end{gathered}$$

which is true because, being $a^2 = b^2 + c^2$, we have

$$2a^2 = 2b^2 + 2c^2 \geq b^2 + c^2 + 2bc = (b+c)^2$$

The equality holds if and only if $b = c$, i.e. if $\triangle ABC$ is an isosceles right-angled triangle. □