Problema J193. Let $ABCD$ be a square of center O. The parallel through O to AD intersects AB and CD at M and N and a parallel to AB intersects diagonal AC at P. Prove that

$$OP^4 + \left(\frac{MN}{2}\right)^4 = MP^2 \cdot NP^2$$

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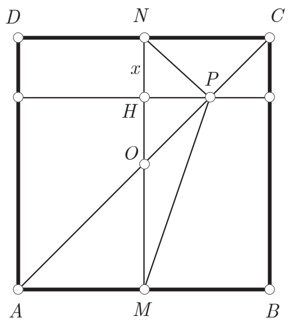
Let H be the intersection point of MN and the parallel to AB through P. By setting $AB = 2a$ and $NH = x$, we have $OH = a - x$, $MH = 2a - x$ (see figure).



Since $PH = OH = a - x$ the Pythagora's theorem yields

$$NP^2 = x^2 + (a-x)^2, \qquad MP^2 = (2a-x)^2 + (a-x)^2 \tag{1}$$

Now, by using (1) and the Apollonius' theorem we have

$$\begin{aligned} OP^2 &= \frac{1}{4}\left(2NP^2 + 2MP^2 - MN^2\right) = \\ &= \frac{1}{4}\left[2x^2 + 2(a-x)^2 + 2(2a-x)^2 + 2(a-x)^2 - 4a^2\right] = \\ &= 2x^2 - 4ax + 2a^2 = 2(a-x)^2 \end{aligned}$$

Therefore

$$\begin{aligned} &OP^4 + \left(\frac{MN}{2}\right)^4 - MP^2 \cdot NP^2 = \\ &= 4(a-x)^4 + a^4 - \left[(2a-x)^2 + (a-x)^2\right]\left[x^2 + (a-x)^2\right] = 0 \end{aligned}$$

and we are done. □