Problema J194. Let a, b, c be the side-lenghts of a triangle with the largest side c. Prove that

$$\frac{ab(2c+a+b)}{(a+c)(b+c)} \leq \frac{a+b+c}{3}$$

Proposed by Arkady Alt, San Jose, California, USA

Solution by Ercole Suppa, Teramo, Italy

The inequality is equivalent to

$$(a+b+c)(a+c)(b+c) - 3ab(2c+a+b) \geq 0$$

Now, bearing in mind that $c \geq a$, $c \geq b$ and $c \geq \frac{a+b}{2}$, we get

$$\begin{aligned}
LHS &= (a+b+c)(a+c)(b+c) - 3ab(2c+a+b) \\
&= (a+b+c)\left(ab+ac+bc+c^2\right) - 3ab(a+b+c) - 3abc \\
&\geq (a+b+c)\left(3ab+c^2\right) - 3ab(a+b+c) - 3abc \\
&= (a+b+c)c^2 - 3abc \\
&= c\left[(a+b+c)c - 3ab\right] \\
&\geq c\left[\left(a+b+\frac{a+b}{2}\right)\frac{a+b}{2} - 3ab\right] \\
&= c\left[\frac{3}{4}(a+b)^2 - 3ab\right] \\
&= \frac{3}{4}c(a-b)^2 \geq 0
\end{aligned}$$

and the proof is complete. □