Problema J195. Find all primes p and q such that both $pq - 555p$ and $pq + 555q$ are perfect squares.

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Since $pq - 555p = p(q - 555)$ is a perfect square, p divides $q - 555$ and $q > 555$. Therefore there exists an integer $a \geq 1$ such that

$$q - 555 = ap \tag{1}$$

Likewise q divides $p + 555$, so there exists an integer $b \geq 1$ such that

$$p + 555 = bq \tag{2}$$

From (1) and (2) it follows that

$$\begin{aligned} & p + 555 = b(555 + ap) \quad \Rightarrow \\ & (1 - ab)p = 555(b - 1) \geq 0 \quad \Rightarrow \\ & 1 - ab \geq 0 \quad \Rightarrow \quad a = 1,\ b = 1 \end{aligned}$$

Therefore $q - p = 555$, so $p = 2$ (otherwise $q - p$ would be an even number) and $q = 557$. □