Problema J196. Let I be the incenter of triangle ABC and let A', B', C' be the feet of altitudes from vertices A, B, C. If $IA' = IB' = IC'$, then prove that triangle ABC is equilateral.

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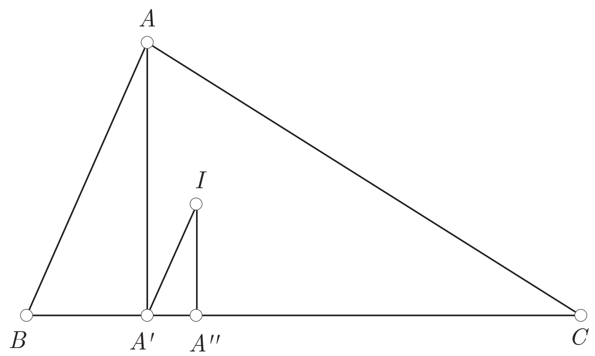
Let $a = BC$, $b = CA$, $c = AB$, $s = (a+b+c)/2$ and let A'', B'', C'' be the feet of the perpendiculars from I to BC, CA, AB respectively.



We have $A'A'' = |BA'' - BA'|$ so, by using the known formulas $BA' = c\cos B$, $BA'' = s - b$ (assuming wlog $a \geq b \geq c$) we have

$$A'A'' = |s - b - c\cos B| = \left| s - b - \frac{a^2+c^2-b^2}{2a} \right| = \frac{b+c-a}{2a}(b-c) \qquad (1)$$

Likewise we obtain

$$B'B'' = |s - a - c\cos A| = \left| s - a - \frac{b^2+c^2-a^2}{2b} \right| = \frac{a+c-b}{2b}(a-c) \qquad (2)$$

(observe that if $A > 90^\circ$ then $AB' = c\cos(180^\circ - A) = -c\cos A$ and $B'B'' = B'A + AB'' = -c\cos A + s - a$).

Since $IA' = IB'$ the triangles $\triangle IA'A''$ and $\triangle IB'B''$ are congruent, so $A'A'' = B'B''$. Hence from (1) and (2) it follows that

$$\frac{b+c-a}{2a}(b-c) = \frac{a+c-b}{2b}(a-c) \qquad \Rightarrow$$

$$b(b+c-a)(b-c) = a(a+c-b)(a-c) \qquad \Rightarrow$$

$$a^3 - b^3 + ab^2 - a^2b + bc^2 - ac^2 \qquad \Rightarrow$$

$$(a-b)(a^2+b^2-c^2) = 0$$

Therefore $(a-b)\cos C = 0$, and analogously we obtain $(c-a)\cos B = 0$. This implies $a = b = c$ because the angles B, C are acute-angled (thanks to our assumption $a \geq b \geq c$). $\square$