Problema J197. Let x, y, z be positive real numbers. Prove that

$$\sqrt{2\left(x^2y^2+y^2z^2+z^2x^2\right)\left(\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}\right)} \geq x\sqrt{\frac{1}{y}+\frac{1}{z}}+y\sqrt{\frac{1}{z}+\frac{1}{x}}+z\sqrt{\frac{1}{x}+\frac{1}{y}}$$

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By Cauchy-Schwarz we have

$$\left(x\sqrt{\frac{1}{y}+\frac{1}{z}}+y\sqrt{\frac{1}{z}+\frac{1}{x}}+z\sqrt{\frac{1}{x}+\frac{1}{y}}\right)^2 \leq 2\left(x^2+y^2+z^2\right)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

In order to complete the proof it is enough to prove that

$$\left(x^2+y^2+z^2\right)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \leq \left(x^2y^2+y^2z^2+z^2x^2\right)\left(\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}\right)$$

The latter inequality is equivalent to

$$\frac{x^5y^5+x^5z^5+y^5z^5-x^4y^3z^3-x^3y^4z^3-x^3y^3z^4}{x^3y^3z^3} \geq 0 \quad \Leftrightarrow$$

$$x^5y^5+x^5z^5+y^5z^5 \geq x^4y^3z^3+x^3y^4z^3+x^3y^3z^4 \quad \Leftrightarrow$$

$$\sum_{\text{sym}} x^5y^5 \geq \sum_{\text{sym}} x^4y^3z^3$$

which is true by Muirhead's inequality, as $(5,5,0) \succ (4,3,3)$. Therefore the conclusion follows. $\square$