Problema J198. Find all pairs (x, y) for which $x! + y! + 3$ is a perfect cube.

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Because our expression is symmetric, without loss of generality we can assume $x \geq y$. Consider the following cases:

(a) If $x \leq 6$, we can easily verify that $x! + y! + 3$ is a perfect cube only when $(x, y) = (5, 2)$ and $(x, y) = (6, 3)$

x/y	0	1	2	3	4	5	6
0	5	5	6	10	28	124	724
1	5	5	6	10	28	124	724
2	6	6	7	11	29	125	725
3	10	10	11	15	33	129	729
4	28	28	29	33	51	147	747
5	124	124	125	129	147	243	843
6	724	724	725	729	747	843	1443

(b) If $x > 6$, $y \in \{0, 1, 2, 5, 6\}$, then $x! + y! + 3 \equiv 3, 4, 5, 6 \pmod 9$. Therefore $x! + y! + 3$ cannot be a perfect cube, since $z^3 \equiv 0, 1, 8 \pmod 9$ for every $z \in \mathbb{N}$.

(c) If $x > 6$, $y = 3$, we have three subcases:

- If $x = 7$, $y = 3$ then $x! + y! + 3 = 5049$ which is not a perfect cube.
- If $x = 8$, $y = 3$ then $x! + y! + 3 = 40329$ which is not a perfect cube.
- If $x \geq 9$, $y = 3$ then $x! + y! + 3 = x! + 9$ which is not a cube since $3 \mid x! + 9$, whereas $27 \nmid x! + 9$.

(d) If $x > 6$, $y = 4$, we have three subcases:

- If $x = 7$, $y = 4$ then $x! + y! + 3 = 5067$ which is not a perfect cube.
- If $x = 8$, $y = 4$ then $x! + y! + 3 = 40347$ which is not a perfect cube.
- If $x \geq 9$, $y = 4$ then $x! + y! + 3 = x! + 27$. It is easy to proof by induction that $x! + 27 > x^3$ for every $x \geq 5$. Therefore if were $x! + 27 = z^3$ we should have $x > z$ and consequently $z = 3^k$ with $k > 1$. Hence $x! + 27 = 3^{3k}$, but this is absurd since $81 \mid 3^{3k}$ whereas $81 \nmid x! + 27$.

(e) If $x > 6$ and $y > 6$ then $x! + y! + 3$ cannot be a perfect cube since $x! + y! + 3 \equiv 3 \pmod 9$.

Finally $x!+y!+3$ is a perfect cube only for $(x, y) \in \{(5, 2), (2, 5), (6, 3), (3, 6)\}$.

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