Problem 363. Extend side CB of triangle ABC beyond B to a point D such that $DB = AB$. Let M be the midpoint of side AC. Let the bisector of $\angle ABC$ intersect line DM at P. Prove that $\angle BAP = \angle ACB$.

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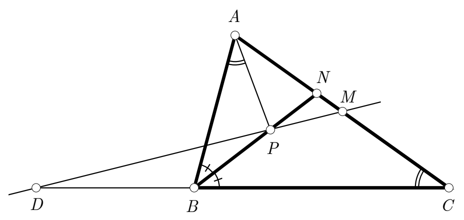

Let $BC = a$, $CA = b$, $AB = c$. Since BN is the bisector of $\angle ABC$ we have $AN = bc/(a + c)$, so

$$NM = AM - AN = \frac{b}{2} - \frac{bc}{a+c} = \frac{b(a-c)}{2(a+c)}$$

Considering line DM cutting $\triangle BCN$, by Menelaus' theorem we get

$$\frac{BP}{PN} \cdot \frac{NM}{MC} \cdot \frac{CD}{BD} = 1 \qquad \Rightarrow$$

$$\frac{BP}{PN} = \frac{MC \cdot BD}{NM \cdot CD} = \frac{\frac{b}{2} \cdot c}{\frac{b(a-c)}{2(a+c)} \cdot (a+c)} = \frac{c}{a-c}$$

Thus $(a - c) \cdot BP = c \cdot PN$ and this implies

$$(a-c) \cdot BP = c \cdot (BN - BP) \qquad \Rightarrow$$

$$a \cdot BP = c \cdot BN \qquad \Rightarrow \qquad \frac{BP}{AB} = \frac{BN}{BC} \qquad (*)$$

From (*), bearing in mind that $\angle ABP = \angle NBC$, it follows that $\triangle ABP$ and $\triangle CBN$ are similar. Therefore $\angle BAP = \angle NCB = \angle ACB$ and claim is proved. □