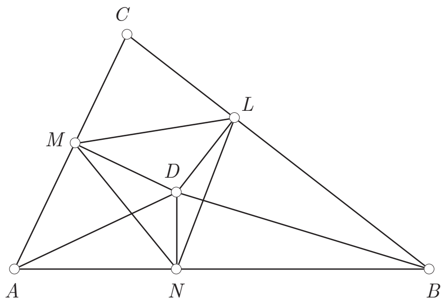
Problema O195. Let O, I, H be the circumcenter, incenter, and orthocenter of a triangle ABC, and let D be an interior point to triangle ABC such that $BC \cdot DA = CA \cdot DB = AB \cdot DC$. Prove that A, B, D, O, I, H are concyclic if and only if $\angle C = 60°$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Ercole Suppa, Teramo, Italy

First of all, observe that D lies on the A-Apollonian circle of $\triangle ABC$ since $\frac{DB}{DC} = \frac{AB}{AC}$. Likewise D lies on the B-Apollonian circle. Hence D is the first point isodynamic, being interior to $\triangle ABC$.

Let $\triangle LMN$ be the pedal triangle of D and let R be the circumradius of $\triangle ABC$ (see figure).



From cyclic quadrilaterals $ANDM$, $BLDN$ we have

$$MN = DA \cdot \sin A, \quad NL = DB \cdot \sin B \quad \Rightarrow$$

$$\frac{MN}{NL} = \frac{DA}{DB} \cdot \frac{\sin A}{\sin B} = \frac{CA}{CB} \cdot \frac{\sin A}{\sin B} = \frac{2R \sin B}{2R \sin A} \cdot \frac{\sin A}{\sin B} = 1$$

Therefore $MN = NL$ and similarly we obtain $NL = LM$, so $\triangle LMN$ is an equilateral triangle. Thus we have

$$\begin{aligned}\angle ADB &= 360° - \angle MDA - \angle LDM - \angle BDL = \\ &= 360° - \angle MNA - (180° - C) - \angle BNL = \\ &= 180° + C - (180° - \angle MNL) = \\ &= 180° + C - 120° = 60° + C\end{aligned}$$

Now it is easy to prove the result.

If A, B, D, O, I, H are concyclic then

$$\angle AIB = \angle ADB \quad \Rightarrow \quad 90° + \frac{C}{2} = 60° + C \quad \Rightarrow \quad C = 60°$$

Conversely if $C = 60°$ we have

$$\begin{aligned}\angle ADB &= 60° + C = 120° \\ \angle AOB &= 2C = 120° \\ \angle AIB &= 90° + \frac{C}{2} = 120° \\ \angle AHB &= A + B = 180° - C = 120°\end{aligned}$$

so A, B, D, O, I, H are concyclic, and we are done. $\square$