Problema O99. Let AB be a chord that is not a diameter of circle ω. Let T be a mobile point on AB. Construct circles ω_1 and ω_2 that are externally tangent to each other at T and internally tangent to ω at T_1 and T_2, respectively. Let $X_1 \in AT_1 \cap TT_2$ and $X_2 \in AT_2 \cap TT_1$. Prove that X_1X_2 passes through a fixed point.

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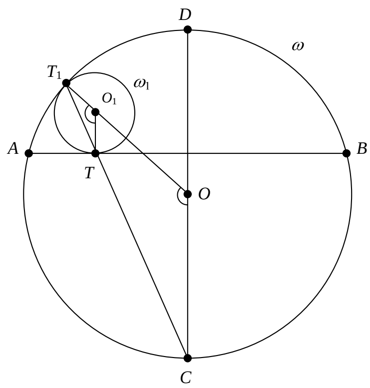
In the following proof we'll use two lemmata:

LEMMA 1. Let AB be a chord that is not a diameter of circle ω, let T be a point on AB, let ω_1 be a circle internally tangent to ω at T_1 and tangent to AB to the point T, let C, D be the intersection points of ω with the perpendicular bisector of AB, with C and T_1 lying on opposite sides of AB. The points C, T, T_1 are collinear.

Proof.

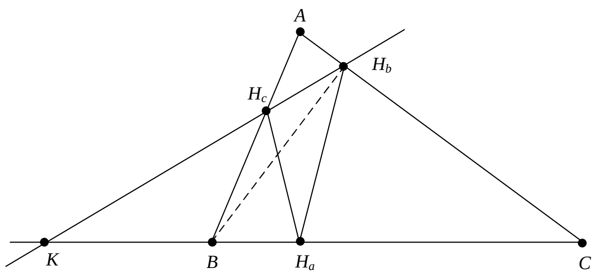
Let O_1 be the center of ω_1. We have $TO_1 \| CD$ because TO_1 and CD are both perpendicular to AB. The isosceles triangles ΔT_1O_1T and ΔT_1OC are similars because $\angle T_1O_1T = \angle T_1OC$. Thus $\angle O_1TT_1 = \angle OCT_1$ and this implies that C, T, T_1 are collinears. ■



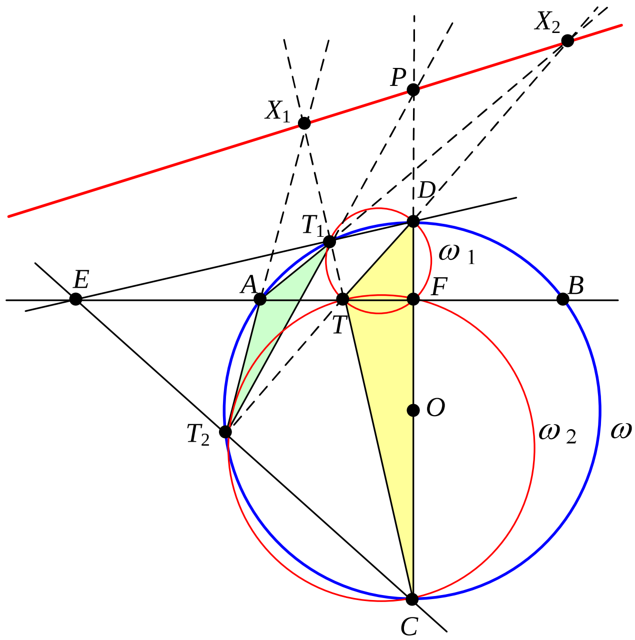
LEMMA 2. Let $H_aH_bH_c$ be the orthic triangle of $\triangle ABC$ and let K be the intersection point of H_bH_c with the line BC. The point K is the harmonic conjugate of H_a with respect to B and C.

Proof.

We suppose, without loss of generality, that $c < b$. Since in $\triangle KH_bH_a$ the lines BH_b, AC are the internal and external angle bisectors, the points K and H_a are harmonic conjugates with respect to B and C. ■



Now we can proof that all the lines X_1X_2 passes through a fixed point.



Let C, D be the intersection points of ω with the perpendicular bisector of AB (with C and T_1 on the opposite sides of AB) and let F be the middle point of AB. We have:

- by LEMMA 1 the point C, T, T_1 are collinears; similarly the point D, T, T_2 are collinears;
- T is the orthocenter of triangle $\triangle CDE$;
- the quadrilateral $TFDT_1$ is cyclic because $CT_1 \perp T_1D$ and $EF \perp FD$; similarly the quadrilateral $CFTT_2$ is cyclic; denote with ω_1, ω_2 the circumcircles of $TFDT_1$ and $CFTT_2$;
- the lines CT_1, BA, DT_2 concur in the point E, radical center of three circles (ABC), (TCT_1), (TDT_2);
- thus the triangles AT_1T_2 , CTD are perspective. Hence based on the Desarques theorem, we conclude that the points $X_1 = AT_2 \cap TT_1$, $P = T_1T_2 \cap CD$, $X_2 = AT_1 \cap TT_2$ are collinear;
- by LEMMA 2 the point P is harmonic conjugate of F with respect to the points C and D, i.e. P is the pole of the line AB wrt the circle ω.

Then P is a fixed point independent from the choice of the point T and the proof is completed. $\square$