Problema S102. Consider triangle ABC with circumcenter O and incenter I. Let E and F be the points of tangency of the incircle with AC and AB, respectively. Prove that EF, BC, OI are concurrent if and only if $r_a^2 = r_b r_c$, where r_a, r_b, r_c are the radii of the excircles.

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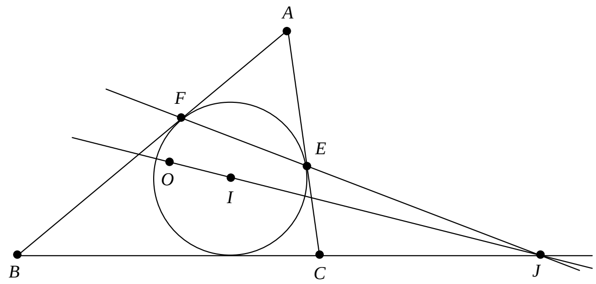

We will use homegeneous barycentric coordinates with respect $\triangle ABC$. Denote as usual by a, b, c,s,Δ the sides BC, CA, AB, the semiperimeter and the area of the triangle respectively. We have:

$$E(s-c:0:s-a) \quad , \quad F(s-b:s-a:0) \quad , \quad I(a:b:c)$$
$$O\left(a^2\left(-a^2+b^2+c^2\right), b^2\left(a^2-b^2+c^2\right), c^2\left(a^2+b^2-c^2\right)\right)$$

Hence the equations of the lines EF, IO are:

$$\begin{aligned} EF &\quad : \quad (s-a)x-(s-b)y-(s-c)z=0 \\ IO &\quad : \quad bc(b-c)(s-a)x+ac(c-a)(s-b)y+ab(a-b)(s-c)z=0 \end{aligned}$$

The equation of the line BC is $x=0$. Therefore the lines EF, IO, BC are concurrent if and only if:

$$\begin{vmatrix} s-a & b-s & c-s \\ bc(b-c)(s-a) & ac(c-a)(s-b) & ab(a-b)(s-c) \\ 1 & 0 & 0 \end{vmatrix} = 0 \qquad \Longleftrightarrow$$

$$a(a+b-c)(a-b+c)\left(b^2+c^2-ab-ac\right)=0$$

On the other hand we have:

$$\begin{aligned} r_a^2 - r_b r_c &= \frac{\Delta^2}{(s-a)^2} - \frac{\Delta^2}{(s-b)(s-c)} = \\ &= \frac{s(s-b)(s-c)}{s-a} - s(s-a) = \\ &= \frac{(a+b+c)\left(b^2+c^2-ab-ac\right)}{2(a-b-c)} \end{aligned}$$

and thus we are done. $\square$