**Problema S185.** Let $A_1$, $A_2$, $A_3$ be non-collinear points on parabola $x^2 = 4py$, $p > 0$, and let $B_1 = \ell_2 \cap \ell_3$, $B_2 = \ell_3 \cap \ell_1$, $B_3 = \ell_1 \cap \ell_2$ where $\ell_1$, $\ell_2$, $\ell_3$ are tangents to the parabola at points $A_1$, $A_2$, $A_3$, respectively. Prove that $[A_1A_2A_3]/[B_1B_2B_3]$ is a costant and find its value.

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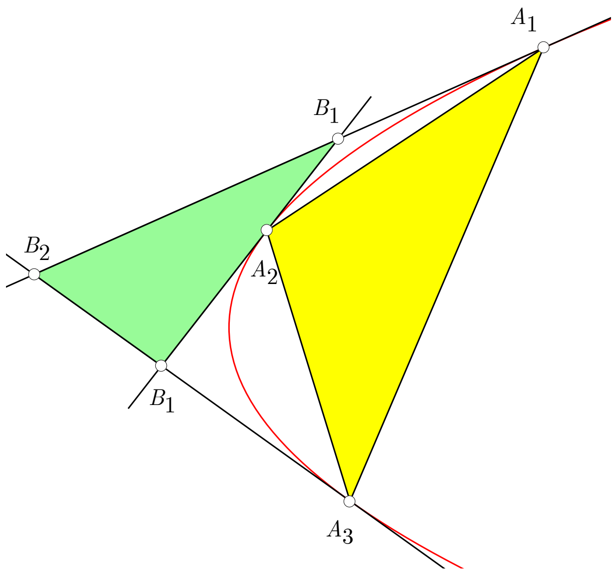
Let $A_1 = \left(u, \frac{1}{4p}u^2\right)$, $A_2 = \left(v, \frac{1}{4p}v^2\right)$, $A_3 = \left(w, \frac{1}{4p}w^2\right)$.




Observe that the equation of the tangent to the parabola $x^2 = 4py$ at its point $T\left(t, \frac{1}{4p}t^2\right)$ is

$$2t(x-t) - 4p\left(y - \frac{1}{4p}t^2\right) = 0 \qquad \Leftrightarrow \qquad 2tx - 4py - t^2 = 0$$

Therefore the equations of the lines $\ell_1$, $\ell_2$, $\ell_3$ are:

$$\begin{aligned} \ell_1 &: 2ux - 4py - u^2 = 0 \\ \ell_2 &: 2vx - 4py - v^2 = 0 \\ \ell_1 &: 2wx - 4py - w^2 = 0 \end{aligned}$$

After some algebra we get

$$B_1 = \left(\frac{v+w}{2}, \frac{vw}{4p}\right), \qquad B_2 = \left(\frac{u+w}{2}, \frac{uw}{4p}\right), \qquad B_3 = \left(\frac{u+v}{2}, \frac{uv}{4p}\right)$$

$$[A_1A_2A_3] = \frac{1}{2}\left|\det\begin{pmatrix} u & \frac{1}{4p}u^2 & 1 \\ v & \frac{1}{4p}v^2 & 1 \\ w & \frac{1}{4p}w^2 & 1 \end{pmatrix}\right| = \frac{1}{8}\left|\frac{(u-v)(u-w)(v-w)}{p}\right| \qquad (*)$$

$$[B_1B_2B_3] = \frac{1}{2}\left|\det\begin{pmatrix} \frac{v+w}{2} & \frac{vw}{4p} & 1 \\ \frac{u+w}{2} & \frac{uw}{4p} & 1 \\ \frac{u+v}{2} & \frac{uv}{4p} & 1 \end{pmatrix}\right| = \frac{1}{16}\left|\frac{(u-v)(u-w)(v-w)}{p}\right| \quad (**)$$

Finally, by using (*),(**) we obtain $\frac{[A_1A_2A_3]}{[B_1B_2B_3]} = 2$, establishing the result. □