Problema S195. Let ABC be a triangle with incenter I and circumcenter O and let M be the midpoint of BC. The bisector of angle A intersects lines BC and OM at L and Q, respectively. Prove that

$$AI \cdot LQ = IL \cdot IQ$$

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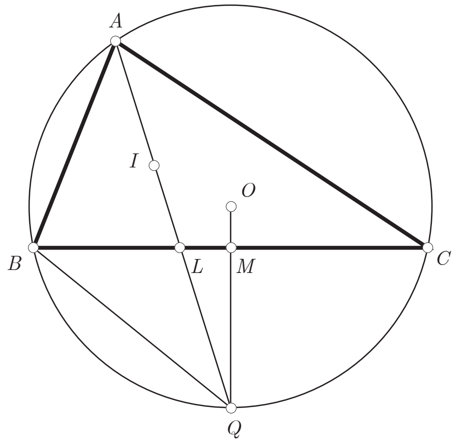
Denote as usual by a, b, c the sides of the triangle opposite to vertexes A, B, C, respectively, and by s the semiperimeter of the triangle.

Notice that AI and OM bisect the arc BC of circumcircle (O) which not contains A. Therefore $Q \in (O)$ (see figure).



We have

$$\frac{AI}{IQ} = \frac{AB}{BQ} \cdot \frac{\sin \frac{B}{2}}{\sin\left(\frac{B}{2} + \frac{A}{2}\right)} = \frac{AB}{BQ} \cdot \frac{\sin \frac{B}{2}}{\cos \frac{C}{2}} \tag{1}$$

and

$$\frac{IL}{LQ} = \frac{BI}{BQ} \cdot \frac{\sin \frac{B}{2}}{\sin \frac{A}{2}} \tag{2}$$

From (1) and (2) it follows that

$$AI \cdot LQ = IQ \cdot \frac{AB}{BQ} \cdot \frac{\sin \frac{B}{2}}{\cos \frac{C}{2}} \cdot IL \cdot \frac{BQ}{BI} \cdot \frac{\sin \frac{A}{2}}{\sin \frac{B}{2}} = IL \cdot IQ \cdot \frac{AB}{BI} \cdot \frac{\sin \frac{A}{2}}{\cos \frac{C}{2}} \tag{3}$$

Using the known formulas

$$BI = \sqrt{\frac{ac(s-b)}{s}} \quad , \quad \sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \quad , \quad \cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

we get

$$\frac{AB}{BI} \cdot \frac{\sin \frac{A}{2}}{\cos \frac{C}{2}} = c\sqrt{\frac{s}{ac(s-b)}}\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{ab}{s(s-c)}} = 1 \tag{4}$$

From (3) and (4) it follows that $AI \cdot LQ = IL \cdot IQ$ and we are done. □