Problema 618.

Sea ABC un triángulo rectángulo en A con catetos c(fijo) y b (variable), y con hipotenusa a variable. Definimos h_a, w_a, m_a, la altura, la bisectriz y la mediana correspondiente a la hipotenusa a. Sean m_b y m_c las medianas correspondientes a los catetos b y c, respectivamente. Calcular los siguientes límites:

(i) $\lim_{b\to c} \dfrac{\sqrt{\frac{1}{b^2}+\frac{1}{c^2}}-\frac{1}{m_a}}{(b-c)^2}$;

(ii) $\lim_{b\to c} \dfrac{\sqrt{\frac{1}{b^2}+\frac{1}{c^2}}-\frac{1}{w_a}}{(b-c)^2}$;

(iii) $\lim_{b\to c} \dfrac{\sqrt{m_b^2+m_c^2}-\sqrt{5}h_a}{(b-c)^2}$;

(iv) $\lim_{b\to c} \dfrac{\sqrt{m_b^2+m_c^2}-\sqrt{5}w_a}{(b-c)^2}$;

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Soluzione di Ercole Suppa.

Essendo $a^2 = b^2 + c^2$ (teorema di Pitagora), dalle note formule di geometria del triangolo con semplici calcoli otteniamo che

$$m_a = \frac{\sqrt{b^2+c^2}}{2}, \qquad m_b = \frac{\sqrt{b^2+4c^2}}{2}, \qquad m_c = \frac{\sqrt{4b^2+c^2}}{2} \tag{1}$$

$$h_a = \frac{bc}{\sqrt{b^2+c^2}}, \qquad w_a = \frac{\sqrt{2}bc}{b+c} \tag{2}$$

(i) Tenuto conto di (1) abbiamo:

$$\begin{aligned}
\lim_{b\to c} \frac{\sqrt{\frac{1}{b^2}+\frac{1}{c^2}}-\frac{1}{m_a}}{(b-c)^2} &= \lim_{b\to c} \frac{\frac{\sqrt{b^2+c^2}}{bc}-\frac{2}{\sqrt{b^2+c^2}}}{(b-c)^2} = \\
&= \lim_{b\to c} \frac{b^2+c^2-2bc}{bc(b-c)^2\sqrt{b^2+c^2}} = \\
&= \lim_{b\to c} \frac{1}{bc\sqrt{b^2+c^2}} = \frac{1}{\sqrt{2}c^3}
\end{aligned}$$

(ii) Tenuto conto di (2) abbiamo:

$$\begin{aligned}\lim_{b\to c}\frac{\sqrt{\frac{1}{b^2}+\frac{1}{c^2}}-\frac{1}{w_a}}{(b-c)^2}&=\lim_{b\to c}\frac{\frac{\sqrt{b^2+c^2}}{bc}-\frac{b+c}{\sqrt{2}bc}}{(b-c)^2}=\lim_{b\to c}\frac{\sqrt{2b^2+2c^2}-b-c}{\sqrt{2}bc(b-c)^2}=\\&=\lim_{b\to c}\frac{b^2+c^2-2bc}{\sqrt{2}bc(b-c)^2\left(\sqrt{2b^2+2c^2}+b+c\right)}=\\&=\lim_{b\to c}\frac{1}{\sqrt{2}bc\left(\sqrt{2b^2+2c^2}+b+c\right)}=\frac{1}{4\sqrt{2}c^3}\end{aligned}$$

(iii) Tenuto conto di (2) abbiamo:

$$\begin{aligned}\lim_{b\to c}\frac{\sqrt{m_b^2+m_c^2}-\sqrt{5}h_a}{(b-c)^2}&=\lim_{b\to c}\frac{\frac{\sqrt{5b^2+5c^2}}{2}-\frac{\sqrt{5}bc}{\sqrt{b^2+c^2}}}{(b-c)^2}=\\&=\lim_{b\to c}\frac{\sqrt{5}\left(b^2+c^2-2bc\right)}{2(b-c)^2\sqrt{b^2+c^2}}=\\&=\lim_{b\to c}\frac{\sqrt{5}}{2\sqrt{b^2+c^2}}=\frac{\sqrt{10}}{4c}\end{aligned}$$

(iv) Tenuto conto di (2) abbiamo:

$$\begin{aligned}\lim_{b\to c}\frac{\sqrt{m_b^2+m_c^2}-\sqrt{5}w_a}{(b-c)^2}&=\lim_{b\to c}\frac{\frac{\sqrt{5b^2+5c^2}}{2}-\frac{\sqrt{10}bc}{b+c}}{(b-c)^2}=\\&=\lim_{b\to c}\frac{(b+c)\sqrt{b^2+c^2}-2\sqrt{2}bc}{2(b+c)(b-c)^2\,(a)}\sqrt{5}=\\&=\lim_{b\to c}\frac{(b+c)^2\left(b^2+c^2\right)-8b^2c^2}{2(b+c)(b-c)^2\left[(b+c)\sqrt{b^2+c^2}+2\sqrt{2}bc\right]}\sqrt{5}=\\&=\lim_{b\to c}\frac{\left(b^2+c^2\right)^2+2bc\left(b^2+c^2\right)+b^2c^2-9b^2c^2}{2(b+c)(b-c)^2\left[(b+c)\sqrt{b^2+c^2}+2\sqrt{2}bc\right]}\sqrt{5}=\\&=\lim_{b\to c}\frac{\left(b^2+c^2+bc\right)^2-9b^2c^2}{2(b+c)(b-c)^2\left[(b+c)\sqrt{b^2+c^2}+2\sqrt{2}bc\right]}\sqrt{5}=\\&=\lim_{b\to c}\frac{\left(b^2+c^2+bc-3bc\right)\left(b^2+c^2+bc+3bc\right)}{2(b+c)(b-c)^2\left[(b+c)\sqrt{b^2+c^2}+2\sqrt{2}bc\right]}\sqrt{5}=\\&=\lim_{b\to c}\frac{(b-c)^2\left(b^2+c^2+4bc\right)}{2(b+c)(b-c)^2\left[(b+c)\sqrt{b^2+c^2}+2\sqrt{2}bc\right]}\sqrt{5}=\\&=\lim_{b\to c}\frac{\left(b^2+c^2+4bc\right)}{2(b+c)\left[(b+c)\sqrt{b^2+c^2}+2\sqrt{2}bc\right]}\sqrt{5}=\\&=\frac{3\sqrt{10}}{16c}\end{aligned}$$