Problema U195. Given a positive integer n, let $f(n)$ be the square of the number of its digits. For example $f(2) = 1$ and $f(123) = 9$. Show that $\sum_{n=1}^{\infty} \frac{1}{nf(n)}$ is convergent.

Proposed by Roberto Bosch Cabrera, Florida, USA

Solution by Ercole Suppa, Teramo, Italy

If k is the the number of digits of the positive integer n, then

$$10^k \leq n < 10^{k+1} \quad \Rightarrow \quad k \leq \log_{10} n < k+1 \quad \Rightarrow \quad k = [\log_{10} n] + 1$$

Therefore $f(n) = ([\log_{10} n] + 1)^2$ and consequently

$$\sum_{n=1}^{\infty} \frac{1}{nf(n)} = \sum_{n=1}^{\infty} \frac{1}{n\left([\log_{10} n] + 1\right)^2} \leq \sum_{n=1}^{\infty} \frac{1}{n\left(\log_{10} n\right)^2} \tag{1}$$

Setting $g(n) = \frac{1}{n(\log_{10} n)^2}$, we have:

$$2^n \cdot g(2^n) = 2^n \cdot \frac{1}{2^n \left(\log_{10} 2^n\right)^2} = \frac{1}{\left(\log_{10} 2\right)^2 n^2} < \frac{1}{n^2}$$

so, by the comparison test, $\sum_{n=1}^{\infty} 2^n g(2^n) < +\infty$ since $\sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty$.

By the the Cauchy condensation test, we get

$$\sum_{n=1}^{\infty} g(n) = \sum_{n=1}^{\infty} \frac{1}{n\left(\log_{10} n\right)^2} < +\infty$$

Finally, according to (1), the given series $\sum_{n=1}^{\infty} \frac{1}{nf(n)}$ converges (by the comparison test). □