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# Rabinowitz Conics Associated with a Triangle

Stanley Rabinowitz[a], Ercole Suppa[b],
Abdilkadir Altintaş[c], and Floor van Lamoen[d]
[a] 545 Elm St Unit 1, Milford, New Hampshire 03055, USA
e-mail: stan.rabinowitz@comcast.net[2]
web: http://www.StanleyRabinowitz.com/
[b] Via B. Croce 54, 64100 Teramo, Italia
e-mail: ercolesuppa@gmail.com
[c] Emirdağ, Afyon, Turkey
e-mail: kadiraltintas1977@gmail.com
[d] Statenhof 3, 4463 TV Goes, The Netherlands
e-mail: fvanlamoen@planet.nl

**Abstract.** Let $P$ be any point in the plane of $\triangle ABC$. Point $U$ is constructed so that vectors $\overrightarrow{AU}$ and $\overrightarrow{BP}$ have the same direction and $AU = AP$. Points $V$, $W$, $X$, $Y$, and $Z$ are constructed in a similar manner. Using computer computation, it was discovered that points $U$, $V$, $W$, $X$, $Y$, and $Z$ lie on a conic. We describe properties of this conic.

**Keywords.** triangle geometry, conic, computer-discovered mathematics, Euclidean geometry, hexagon with opposite sides parallel, Mathematica.

**Mathematics Subject Classification (2020).** 51M04, 51-08.

## 1. Introduction

The main result of this paper is the following.

**Theorem 1** (Existence of Rabinowitz Conic). *Let $P$ be any point in the plane of $\triangle ABC$. Point $U$ is constructed so that vectors $\overrightarrow{AU}$ and $\overrightarrow{BP}$ have the same direction and $AU = AP$. Points $V$, $W$, $X$, $Y$, and $Z$ are constructed in a similar manner, as shown in Figure 1. (Line segments colored the same have the same length.) Then there is a conic that passes through the six points $U$, $V$, $W$, $X$, $Y$, and $Z$.*



[2] Corresponding author

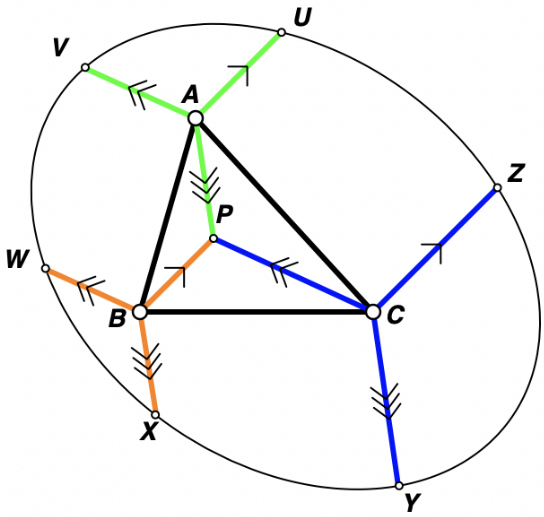


FIGURE 1. conic through six points

Many scholarly papers start off by stating a theorem (with no explanation of where it came from) and then giving a proof. We feel it is equally important to explain how the theorem was found, especially when computer computation is involved.

In section 2, we will describe how the result was found using computer computation. In section 3, we will give a purely geometric proof. In section 4, we will give some related results.

## 2. How the Result was Found

One author (Rabinowitz) was interested in metric properties of well-known points associated with a triangle. He took the first 100 triangle centers cataloged in the Encyclopedia of Triangle Centers [5] and formed the distances between these points and the vertices of the reference triangle. For each center, he examined the distances and checked to see if they satisfied simple relationships, such as $x + y = z$, $wx = yz$, $1/w + 1/x = 1/y + 1/z$, etc. He found the following result which he was not familiar with.

**Theorem 2** (Sum of Squares Property of the de Longchamps Point)**.** *Let $L$ be the de Longchamps point (center $X_{20}$) of $\triangle ABC$ (Figure 2). Then*

$$AB^2 + AL^2 = BC^2 + CL^2,$$
$$BC^2 + BL^2 = CA^2 + AL^2,$$
$$CA^2 + CL^2 = AB^2 + BL^2.$$

This is readily proved using the barycentric coordinates for the de Longchamps point given in [5] and the formula for the distance between two points given their barycentric coordinates.

Rabinowitz published this result in the Facebook group, "Plane Geometry Research" [9]. One of the members of the group, Floor van Lamoen, saw the expressions being the sum of two squares and thought that by perhaps creating right triangles with the appropriate lengths, all that would be needed was to prove

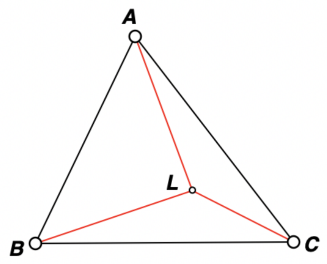


Figure 2. de Longchamps point

that the hypotenuses of these right triangles were equal in pairs. To this end, he erected perpendiculars to the sides of the triangle with the appropriate lengths, as shown in Figure 3. For example, the expression $AB^2+AL^2 = AB^2+AV^2 = BV^2$. Similarly, $BC^2+CL^2 = BC^2+CY^2 = BY^2$. So the first equation in this theorem could be proven geometrically if it could be shown that $BV = BY$. This approach did not pan out, but van Lamoen [7] conjectured from looking at his figure that the points $U$, $V$, $W$, $X$, $Y$, and $Z$ lie on an ellipse.

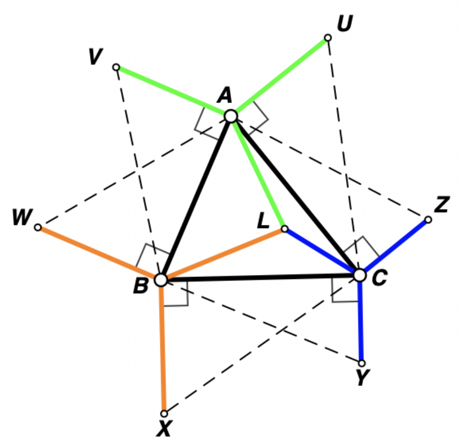


Figure 3.

The authors studied this conjecture and found it to be false, but it suggested that we look for other points for which it might be true.

Altıntaş [1] found that the result was true when $L$ is replaced by the orthocenter of the triangle, provided that the triangle is acute.

**Theorem 3.** *Let $H$ be the orthocenter of acute $\triangle ABC$. Point $U$ is constructed so that $U$ and $B$ are on opposite sides of $AC$, $AU \parallel BH$, $AU \perp AC$, and $AU = AH$. Points $V$, $W$, $X$, $Y$, and $Z$ are constructed in a similar manner, as shown in Figure 4. Then there is a conic that passes through the six points $U$, $V$, $W$, $X$, $Y$, and $Z$.*

Altıntaş found the barycentric equation for the conic. We do not give the equation here because it contains 189 terms.

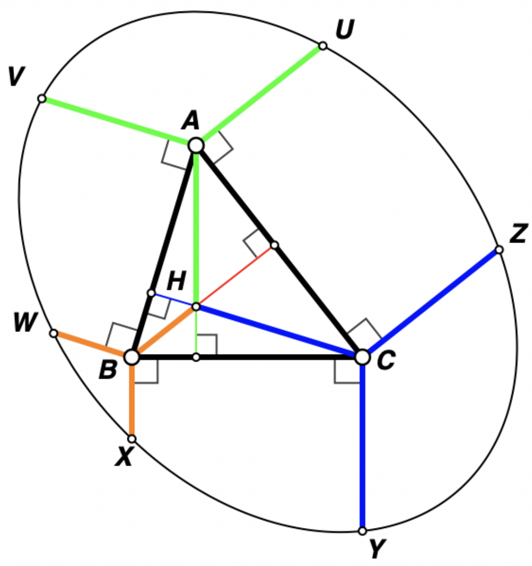


FIGURE 4. conic associated with orthocenter

He also found the center of the conic. The first barycentric coordinate is $-a(a^9 + a^8(b + c) + (b - c)^4(b + c)^3(b^2 + c^2) + 8a^3(b^2 - c^2)^2(b^2 + c^2) - 2a^6(2b^3 + b^2c + bc^2 + 2c^3) + a^5(-6b^4 + 8b^2c^2 - 6c^4) - 3a(b^4 - c^4)^2 + 2a^4(3b^5 + b^3c^2 + b^2c^3 + 3c^5) + 2a^2(-2b^7 + b^6c + b^5c^2 + b^2c^5 + bc^6 - 2c^7$.

We then tried to find other points that had this property. Altıntaş [2] found that the result was not true if $P$ is the circumcenter or the centroid.

Rabinowitz [8] then looked to see what property the orthocenter had that these other points did not have. He noted that the lines through the orthocenter (the altitudes) were perpendicular to the sides and so were the lines erected at the vertices. He suggested that for the centroid, the lines should be parallel to the medians rather than the altitudes. After studying accurate drawings made with Geometer's Sketchpad, he conjectured the following.

**Theorem 4.** *Let $M$ be the centroid of $\triangle ABC$. Points $U$, $V$, $W$, $X$, $Y$, and $Z$ are constructed as in Theorem 1. See Figure 5 in which line segments colored the same have the same length. Then there is an ellipse that passes through the six points $U$, $V$, $W$, $X$, $Y$, and $Z$.*

Suppa [12] then proved that the conjecture was true for all triangles using barycentric coordinates and Mathematica. He found the barycentric equation for the ellipse.

At this point we wanted to know if this result was true for any point $P$.

Finally, Suppa [12] proved that the result was true for all points $P$ and all shape triangles. He proved, using Mathematica and barycentric coordinates, that the six points always lie on a conic. He called the conic a *Rabinowitz $P$-conic*. Here is his proof.

*Proof.* We use barycentric coordinates related to reference triangle $ABC$ with side lengths $a$, $b$, $c$. Let $d_a$, $d_b$, $d_c$ be the distances of $P$ from the vertices $A$, $B$, $C$ respectively. Let the barycentric coordinates of point $P$ be $(u : v : w)$.

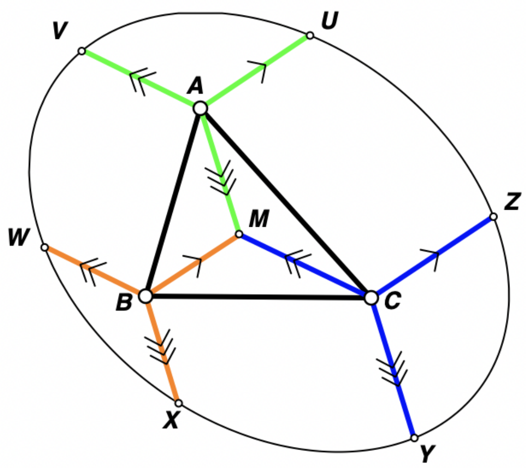


Figure 5. ellipse associated with centroid

The barycentric coordinates of $U$, $V$, $W$, $X$, $Y$, and $Z$ (see Figure 1) are as follows.

$$\begin{aligned}
U &= ((d_a+d_b)u + d_b(v+w) : -d_a(u+w) : d_a w)\\
V &= ((d_a+d_c)u + d_c(v+w) : d_a v : -d_a(u+v))\\
W &= (d_b u : d_c(u+w) + (d_b+d_c)v : -d_b(u+v))\\
X &= (-d_b(v+w) : d_a u + (d_a+d_b)v + d_a w : d_b w)\\
Y &= (-d_c(v+w) : d_c v : d_a u + d_a v + (d_a+d_c)w)\\
Z &= (d_c u : -d_c(u+w) : d_b u + d_b v + (d_b+d_c)w)
\end{aligned}$$

To find the equation of the conic passing through $U$, $V$, $W$, $X$, and $Y$, we first construct the pencil of conics passing through the first four points.

The equations of lines $UV$, $WX$, $VX$, and $UW$ are as follows.

$$\begin{aligned}
UV\ &: d_a ux + (d_a u + d_b u + d_b v + d_c w)y + (d_a u + d_c u + d_b v + d_c w)z = 0\\
WX &: (d_a u + d_a v + d_b v + d_c w)x + d_b vy + (d_a u + d_b v + d_c v + d_c w)z = 0\\
UX\ &: d_a wx + d_b wy - (d_a u + d_b v)z = 0\\
VW\ &: d_a\left(d_a u + (d_a+d_b)v\right)x + d_b(d_a v - d_c w)y\\
&\quad + \left(d_a(d_a+d_c)u + (d_a d_b + d_b d_c + d_a d_c)v + d_a d_c w\right)z = 0
\end{aligned}$$

The equation of the pencil of conics passing through $U$, $V$, $W$, and $X$ is:

$$\Gamma_k : \quad UV \cdot WX + k \cdot UX \cdot VW = 0$$

The conic $\Gamma_k$ passes through $Y$ if $k = 1$. Plugging $k = 1$ into this equation, after some algebra, we obtain the equation of conic passing through $U$, $V$, $W$, $X$, and $Y$:

$$\Gamma(P) : \quad fx^2 + gy^2 + hz^2 + 2pyz + 2qzx + 2rxy = 0$$

where

$$
\begin{aligned}
f =& 2d_a\left(d_a u^2 + (d_a + d_b)uv + d_a vw + (d_a + d_c)uw\right)\\
g =& 2d_b\left(d_b v^2 + (d_a + d_b)uv + (d_b + d_c)vw + d_b uw\right)\\
h =& 2d_c\left(d_c w^2 + d_c uv + (d_b + d_c)vw + (d_a + d_c)uw\right)\\
p =& d_a^2 u^2 + d_b(d_b + d_c)v^2 + d_c(d_b + d_c)w^2 + (2d_a d_b + 2d_b d_c + d_a d_c)uv+\\
&+ (d_b^2 + 4d_b d_c + d_c^2)vw + (d_a d_b + 2d_a d_c + 2d_b d_c)uw\\
q =& d_a(d_a + d_c)u^2 + d_b^2 v^2 + d_c(d_a + d_c)w^2 + (2d_a d_b + 2d_a d_c + d_b d_c)uv+\\
&+ (d_a d_b + 2d_a d_c + 2d_b d_c)vw + (d_a^2 + 4d_a d_c + d_c^2)uw\\
r =& d_a(d_a + d_b)u^2 + d_b(d_a + d_b)v^2 + d_c^2 w^2 + \left(d_a^2 + 4d_a d_b + d_b^2\right)uv+\\
&+ (2d_a d_b + d_a d_c + 2d_b d_c)vw + (2d_a d_b + 2d_a d_c + d_b d_c)\,uw
\end{aligned}
$$

A boring calculation show that $\Gamma(P)$ also passes through point $Z$. $\square$

This proved our main result, Theorem 1.

## 3. A Geometric Proof

The computer computations involved in proving Theorem 1 were quite complicated. We therefore wondered if a simpler geometric proof could be found.

Using Geometer's Sketchpad, Rabinowitz decided that a good start would be to locate the center of the conic. It is well known [10, p. 127] that the center of an ellipse or hyperbola can be found by constructing the intersection of two diameters. A diameter of a conic is a chord that passes through the center of the conic. A diameter can be found by joining the midpoints of two parallel chords [13, p. 82]. Rabinowitz started by drawing a chord through $Y$ that was parallel to the chord through $X$ and $P$. The drawing (Figure 6) revealed two interesting facts, which we now state as a theorem. Both of these facts can be proven using elementary geometry.

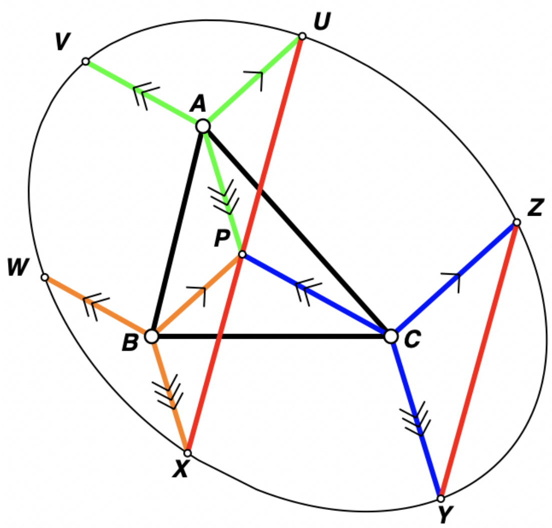


Figure 6.

**Theorem 5.** *Let $P$ be a point in the plane of $\triangle ABC$. Let $U$, $V$, $W$, $X$, $Y$, and $Z$ be the six points associated with the Rabinowitz $P$-conic, as shown in Figure 6. Then chord $XU$ passes through $P$ and chord $YZ$ is parallel to chord $XU$.*

*Proof.* Since $AU \parallel BP$ and $AP \parallel BX$, this implies that $\angle UAP = \angle PBX$. Since both triangles $UAP$ and $PB$X are isosceles, this means that they must be similar and $XP \parallel PU$, which implies that $XPU$ is a straight line.

Since $BP \parallel CZ$ and $BX \parallel CY$, this implies that $\angle PBX = \angle ZCY$. Since both triangles $PBX$ and $ZCY$ are isosceles, this means that they must be similar and $XP \parallel YZ$. □

In the same manner, $XW \parallel YPV$ and $VU \parallel WPZ$. Thus, figure $XUVYZW$ is a (re-entrant) hexagon with its opposite sides parallel. See Figure 7.

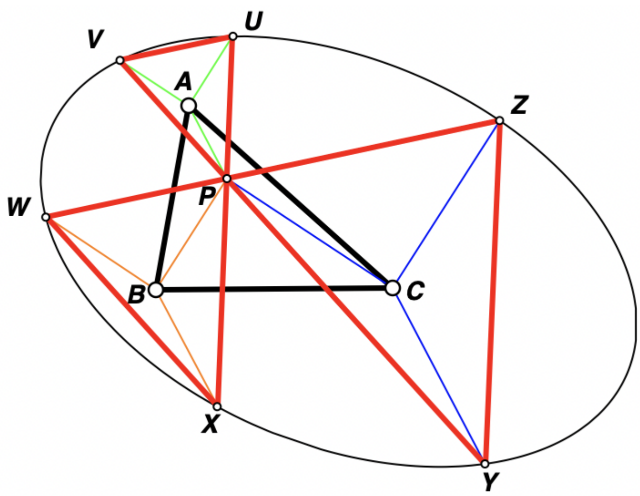


FIGURE 7. hexagon with opposite sides parallel

This gives us a purely geometric proof of Theorem 1.

*Proof of Theorem 1.* Note that $XUVYZW$ is a hexagon with its opposite sides parallel. For such a hexagon, a well-known result states that the vertices of the hexagon must lie on a conic. See Theorem 1 in [4] or [13, p. 281]. □

## 4. ADDITIONAL RESULTS

Suppa [12] determined when the Rabinowitz conic is an ellipse, parabola or hyperbola.

**Theorem 6** (Suppa's Classification of the Rabinowitz Conic)**.** *Let $P$ be any point in the plane of $\triangle ABC$. The points $U$, $V$, $W$, $X$, $Y$, and $Z$ are constructed as in Theorem 1 (Figure 1).*

▷ *Let $\mathcal{C}$ be the conic that passes through the six points $U$, $V$, $W$, $X$, $Y$, and $Z$.*
▷ *Let $d_a$, $d_b$, $d_c$ be the distances from $P$ to the vertices $A$, $B$, and $C$, respectively.*
▷ *Let $D = d_a^2 + d_b^2 + d_c^2 - 2d_ad_b - 2d_ad_c - 2d_bd_c$.*

*Then,*
*if $D > 0$, the conic is a hyperbola,*
*if $D = 0$, the conic is a parabola,*
*if $D < 0$, the conic is an ellipse.*

*Proof.* The discriminant of conic $\Gamma(P)$ is given by

$$\Delta = \frac{1}{4}\left(d_a^2 + d_b^2 + d_c^2 - 2d_ad_b - 2d_bd_c - 2d_ad_c\right)(u+v+w)^2(d_au + d_bv + d_cw)^2.$$

Thus, $\Delta > 0$, $\Delta = 0$, or $\Delta < 0$ depending on whether

$$d_a^2 + d_b^2 + d_c^2 - 2\left(d_ad_b + d_bd_c + d_cd_a\right)$$

is greater than, equal to, or less than 0. ☐

All three cases can actually occur. For example, the Rabinowitz $P$-conic is a hyperbola for a 5–5–1 triangle when $P$ is the orthocenter. If $d_a = d_b$ and $D = 0$, then $d_c = 4d_a$. So the conic is a parabola for an isosceles triangle $ABC$ with $P$ on the altitude to the base $BC$, such that $PA = 4PB$. An example is shown in Figure 8.

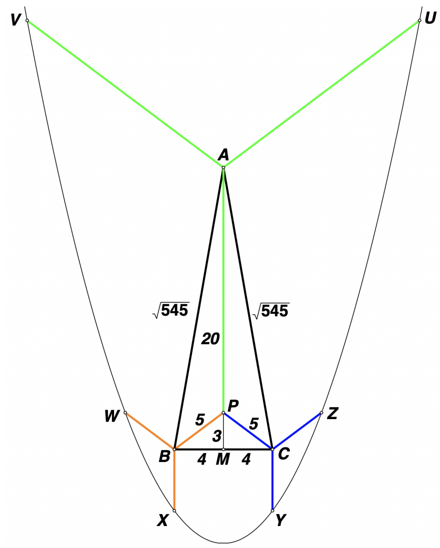


FIGURE 8. parabola

Suppa used Mathematica to study the location of the center of the conic. He found the barycentric coordinates of the center. The first coordinate is

$$ud_a(d_a + d_b + d_c) + (v+w)\left(-d_b^2 - d_c^2 + d_a(d_b + d_c)\right)$$

with similar expressions for the other coordinates.

Using these coordinates, Suppa found that the center of the conic coincided with $P$ when $P$ was the circumcenter of the triangle. Knowing this fact, we were able to come up with a simple geometric proof.

**Theorem 7.** *The center of the Rabinowitz $P$-conic coincides with $P$ if and only if $P$ is the circumcenter of $\triangle ABC$.*

*Proof.* When $P$ is the circumcenter of $\triangle ABC$, $PA = PB$ (Figure 9). Since triangles $PAU$ and $XBP$ are known to be similar, this means that these triangles are congruent. Thus $PU = PX$. Similarly, $PV = PY$ and $PW = PZ$. If two chords of a conic are bisected by their point of intersection, then that point of intersection must be the center of the conic. (See [3, Proposition 43].) Therefore, $P$ is the center of the conic.

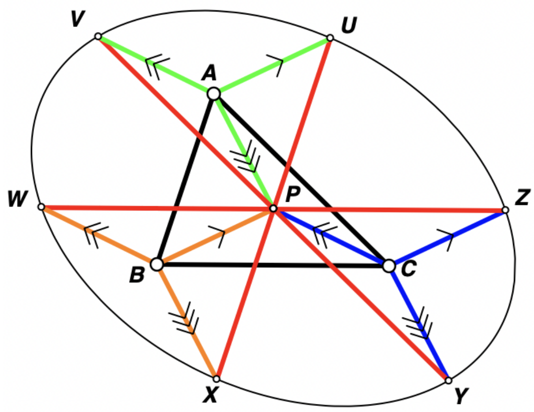


Figure 9.

Conversely, if $P$ is the center of the conic, then $PU = PX$ since all chords through the center of a conic are bisected by the center. Since triangles $PAU$ and $XBP$ are known to be similar, this means that these triangles are congruent. Thus $PA = PB$. Similarly, $PA = PC$ and $P$ is the circumcenter of $\triangle ABC$. □

Floor van Lamoen remarked in [6] that the hexagon formed by the points $U$, $V$, $W$, $X$, $Y$, and $Z$ has an inconic.

**Theorem 8.** *Let $P$ be a point in the plane of $\triangle ABC$. Points $U$, $V$, $W$, $X$, $Y$, and $Z$ are constructed as in Theorem 1. If $U$, $V$, $W$, $X$, $Y$, and $Z$ lie on a conic, then hexagon $UVWXYZ$ also admits an inconic. See Figure 10.*

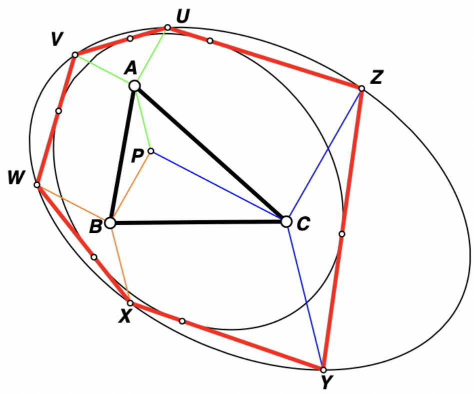


Figure 10. Hexagon with inconic and circumconic

*Proof.* By Theorem 5, $UX$, $VY$, and $WZ$ concur at point $P$. Therefore, hexagon $UVWXYZ$ admits an inconic by the converse of Brianchon's Theorem. □

Recall that the line joining the midpoints of parallel chords in a conic passes through the center of the conic. This gives us a geometric construction for the center.

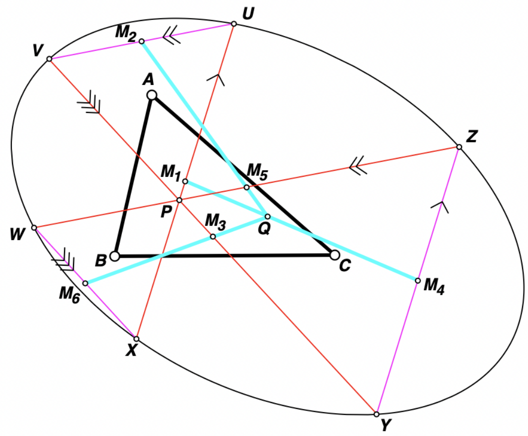


FIGURE 11. Q is center of the conic

**Theorem 9.** *Let $P$ be a point in the plane of $\triangle ABC$. Points $U$, $V$, $W$, $X$, $Y$, and $Z$ are constructed as in Theorem 1. Let $M_1$, $M_2$, $M_3$, $M_4$, $M_5$, and $M_6$ be the midpoints of $XU$, $UV$, $VY$, $YZ$, $ZW$, and $WX$, respectively (Figure 11). Then $M_1M_4$, $M_2M_5$, and $M_3M6$ meet at the center of the Rabinowitz $P$-conic.*

*Proof.* By Theorem 5, $XU \parallel YZ$, $UV \parallel ZW$, and $VY \parallel WX$. Thus $M_1M_4$, $M_2M_5$, and $M_3M_6$ meet at a point, $Q$, which is the center of the conic. □

Vu [14] gave a related result.

**Theorem 10.** *Let $P$ be any point in the plane of $\triangle ABC$. Point $U$ is constructed so that vectors $\overrightarrow{AU}$ and $\overrightarrow{BP}$ have the same direction and $AU = BP$. Points $V$, $W$, $X$, $Y$, and $Z$ are constructed in a similar manner, as shown in Figure 12. (Line segments colored the same have the same length.) Then there is a conic that passes through the six points $U$, $V$, $W$, $X$, $Y$, and $Z$.*

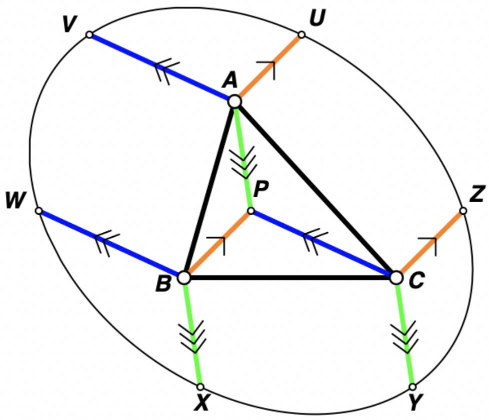


FIGURE 12. Vu's conic

*Proof.* Triangles $UAP$ and $PBX$ are congruent, so $UP = PX$ and $UPX$ is a straight line. Similarly, for $VY$ and $WZ$. The conic through $U$, $V$, $W$, $X$, and $Y$ has two chords ($UX$ and $VY$) bisecting each other at $P$. Therefore $P$ is the center of this conic. Since $P$ also bisects $WZ$, $Z$ must lie on the conic. □

## 5. Areas for future research

**Open Question 1.** *For any triangle $\triangle ABC$, must there exist a point $P$ such that the Rabinowitz $P$-conic is a circle? Is the center of this circle a known triangle center?*

Suppa found a partial result. Using Mathematica, he found that when $\triangle ABC$ is equilateral, such a point $P$ lies at the center of the triangle. Knowing this fact, we were able to come up with a simple geometric proof.

**Theorem 11.** *If $\triangle ABC$ is equilateral with side $a$ and $P$ is the circumcenter of the triangle, then the Rabinowitz $P$-conic is a circle with center $P$ and radius $a$ (Figure 13).*

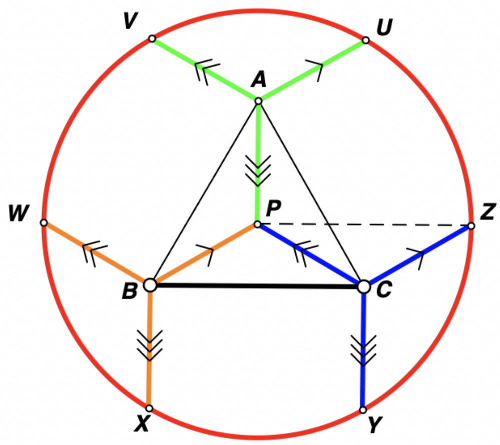


FIGURE 13. equilateral triangle

*Proof.* Let $a$ be the length of a side of the triangle. Since $P$ is the circumcenter of $\triangle ABC$, $PB = PC = CZ$. Since $BP \parallel CZ$, quadrilateral $PBCZ$ is a parallelogram. Therefore, $PZ = BC = a$. Similarly, $PU = PV = PW = PX = PY = a$, and the conic is a circle with center $P$ and radius $a$. $\square$

**Open Question 2.** *When $P$ is some common center, such as the orthocenter, incenter, circumcenter, or centroid, does the Rabinowitz $P$-conic reduce to a known conic? Does it pass through any notable triangle centers? Is the center of the conic a notable point?*

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