3452. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Prove the following and generalize these results.

(a) $\tan^2 36° + \tan^2 72° = 10$,

(b) $\tan^4 36° + \tan^4 72° = 90$,

(c) $\tan^6 36° + \tan^6 72° = 850$,

(d) $\tan^8 36° + \tan^8 72° = 8050$.

Solution by Ercole Suppa, Teramo, Italy

By using the well known results

$$\sin 36° = \frac{1}{4}\sqrt{10 - 2\sqrt{5}}, \qquad \cos 36° = \frac{\sqrt{5}+1}{4}$$

and the double-angle formulas we have

$$a = \tan^2 36° = 5 - 2\sqrt{5}, \qquad b = \tan^2 72° = 5 + 2\sqrt{5}$$

Therefore, taking into account that $a + b = 10$ and $ab = 5$, we obtain

(a) $\tan^2 36° + \tan^2 72° = a + b = 10$,

(b) $\tan^4 36° + \tan^4 72° = a^2 + b^2 = (a + b)^2 - 2ab = 100 - 10 = 90$,

(c) $\tan^6 36° + \tan^6 72° = a^3 + b^3 = (a + b)^3 - 3ab(a + b) = 1000 - 150 = 850$,

(d) $\tan^8 36° + \tan^8 72° = a^4 + b^4 = (a^2 + b^2)^4 - 2a^2b^2 = 8100 - 50 = 8050$

By means of binomial theorem we can generalize these results, as follows

$$x_n = \tan^n 36° + \tan^n 72° = \left(5 - 2\sqrt{5}\right)^n + \left(5 + 2\sqrt{5}\right)^n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k} 2^{2k+1} \cdot 5^{n-k}$$

Alternatively the terms of sequence $x_n = a^n + b^n$ can be calculated by using the recurrence equation

$$x_{n+1} = 10x_n - 5x_{n-1}, \qquad x_1 = 10,\ x_2 = 90$$

□