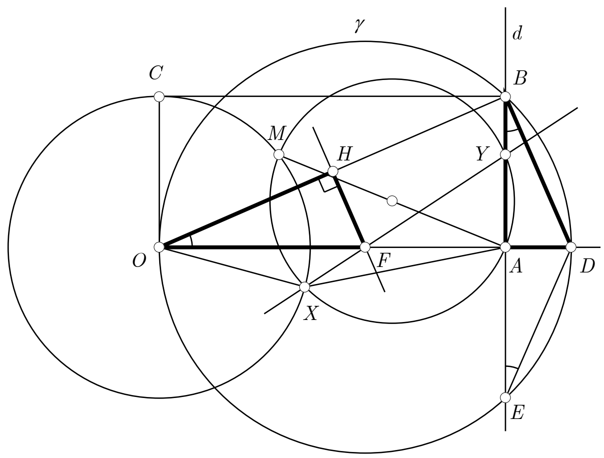
3510. *Proposed by Cosmin Pohoaţă, Tudor Vianu National College, Bucharest , Romania.*

Let d be a line exterior to a given circle Γ with center O. Let A be the orthogonal projection of O on the line d, M be a point on Γ, and X, Y be the intersections of Γ, d with the circle Γ' of diameter AM. Prove that the line XY passes through a fixed point as M moves about Γ.

Solution by Ercole Suppa, Teramo, Italy and Sebastiano Mosca, Pescara, Italy

Let M be an arbitrary point on Γ; let C be the intersection of Γ with the line through O parallel to d (with C and M lying on same side of OA); let B be the orthogonal projection of C on the line d; let X, Y be the intersections of Γ, d with the circle Γ' of diameter AM; let $F = XY \cap OA$; let γ be the circle with center F passing through O; let D be the symmetric of O with respect to point F; let E the symmetric of B with respect to OA; let H the midpoint of OB.



Draw now the perpendicular bisectors of OB, which clearly pass through H and F. Observe that the right triangles $\triangle OHF$ and $\triangle BAD$ are similar, since

$$\angle HOF = \angle BED = \angle ABD$$

Therefore

$$OF : BD = OH : AB \qquad \Rightarrow \qquad OF = \frac{OH \cdot BD}{AB} \tag{1}$$

Denoting with r, a the radius of Γ and the distance from O the the line d, by the Pythagoras and second Euclid's theorems, we have

$$OH = \frac{1}{2}AB = \frac{1}{2}\sqrt{a^2 + r^2}, \qquad AD = \frac{AB^2}{OA} = \frac{r^2}{a} \tag{2}$$

Hence

$$BD = \sqrt{AB^2 + AD^2} = \sqrt{r^2 + \frac{r^4}{a^2}} = \frac{r}{a}\sqrt{a^2 + r^2} \tag{3}$$

From (1), (2) and (3) it follows that

$$OF = \frac{\frac{1}{2}\sqrt{a^2 + r^2} \cdot \frac{r}{a}\sqrt{a^2 + r^2}}{r} = \frac{a^2 + r^2}{2a}$$

and this implies that F is a fixed point of the line OA. The proof is finished. $\square$