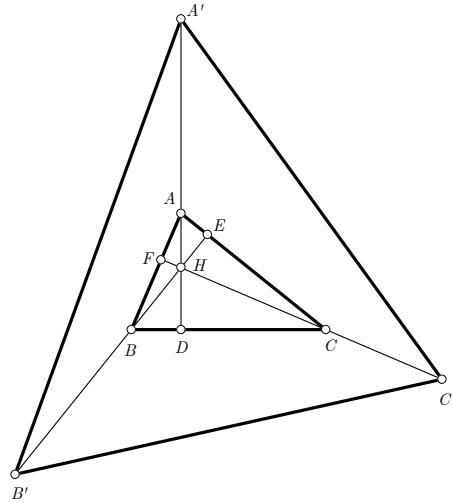


3577. Proposed by Mehmet Sahin, Ankara, Turkey.

Let H be the orthocentre of the acute triangle ABC with a' on the ray HA and such that $AA' = BC$. Define B' , C' similarly. Prove that

$$\text{Area}(A'B'C') = 4\text{Area}(ABC) + \frac{a^2 + b^2 + c^2}{2}$$

Solution by Ercole Suppa, Teramo



Let R , Δ be the circumradius and the area of $\triangle ABC$; let D , E , F be the feet of altitudes onto BC , CA , AB respectively. Bearing in mind the well known formula $AH = 2R \cos A$ and the extended sine law we have

$$A'H = a + 2R \cos A = 2R(\sin A + \cos A)$$

and similar relations hold for $B'H$, $C'H$. Therefore, since

$$\angle A'HB' = \angle AHF + \angle FHB = 90^\circ - B + 90^\circ - A = C$$

we obtain

$$\begin{aligned} \text{Area}(A'B'H) &= \frac{1}{2} A'H \cdot B'H \cdot \sin C = \\ &= 2R^2 (\sin A + \cos A)(\sin B + \cos B) \sin C = \\ &= 2R^2 [\sin A \sin B + \cos A \cos B + \sin(A+B)] \sin C = \\ &= 2R^2 (\sin A \sin B \sin C + \cos A + \cos B \sin C + \sin^2 C) \end{aligned} \quad (1)$$

Hence, by using (1) and its cyclic relations, we get

$$\begin{aligned}
& \text{Area}(A'B'C') = \\
& = \text{Area}(A'B'H) + \text{Area}(B'HC') + \text{Area}(C'HA') = \\
& = 2R^2 \sum_{\text{cyclic}} (\sin A \sin B \sin C + \cos A \cos B \sin C + \sin^2 C) = \\
& = 2R^2 (\sin^2 A + \sin^2 B + \sin^2 C + 3 \sin A \sin B \sin C + \sum_{\text{cyclic}} \cos A \cos B \sin C)
\end{aligned} \tag{2}$$

Now, notice that

$$2R^2 (\sin^2 A + \sin^2 B + \sin^2 C) = \frac{a^2 + b^2 + c^2}{2} \tag{3}$$

$$6R^2 \sin A \sin B \sin C = 6R^2 \cdot \frac{a}{2R} \cdot \frac{b}{2R} \cdot \frac{c}{2R} = \frac{3abc}{4R} = 3\Delta \tag{4}$$

Moreover, by using the projection's theorem $a = b \cos C + c \cos B$, we have

$$\begin{aligned}
& 2R^2 \sum_{\text{cyclic}} \cos A \cos B \sin C = \\
& = 2R^2 (\cos A \cos B \sin C + \cos A \cos C \sin B + \cos B \cos C \sin A) = \\
& = R(c \cos A \cos B + b \cos A \cos C + a \cos B \cos C) = \\
& = R[(c \cos B + b \cos C) \cos A + a \cos B \cos C] = \\
& = aR(\cos A + \cos B \cos C) = \\
& = aR[-\cos(B+C) + \cos B \cos C] = \\
& = aR \sin B \sin C = \\
& = \frac{abc}{4R} = \Delta
\end{aligned} \tag{5}$$

Finally, by plugging (3), (4), (5) into (2) we get the desired relation. \square