Problema 11476. Let a, b and c be the side-lengths of a triangle, and let r be its inradius. Show that

$$\frac{a^2bc}{(b+c)(b+c-a)}+\frac{b^2ca}{(c+a)(c+a-b)}+\frac{c^2ab}{(a+b)(a+b-c)}\geq 18r^2$$

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Solution by Ercole Suppa, Teramo, Italy. Denote by s and Δ the semiperimeter and the area of triangle ABC respectively. By using the well known formulas

$$r=\frac{\Delta}{s} \qquad , \qquad \Delta=\sqrt{s(s-a)(s-b)(s-c)}$$

the desired inequality rewrites as

$$\sum_{\text{cyclic}}\frac{a^2bc}{(b+c)(b+c-a)}\geq\frac{18(s-a)(s-b)(s-c)}{s}$$

Taking into account Padoa's inequality

$$abc\geq 8(s-a)(s-b)(s-c)$$

it is enough to prove that

$$\sum_{\text{cyclic}}\frac{a}{(b+c)(b+c-a)}\geq\frac{9}{2(a+b+c)} \tag{1}$$

Since (1) is symmetric, we can assume $a\leq b\leq c$. Thus

$$\frac{a}{b+c}\leq\frac{b}{a+c}\leq\frac{c}{a+b} \quad , \quad \frac{1}{b+c-a}\leq\frac{1}{a+c-b}\leq\frac{1}{a+b-c}$$

and Chebyshev's inequality yields

$$\sum_{\text{cyclic}}\frac{a}{(b+c)(b+c-a)}\geq\frac{1}{3}\sum_{\text{cyclic}}\frac{a}{b+c}\sum_{\text{cyclic}}\frac{1}{b+c-a} \tag{2}$$

From Nesbitt's and AM-HM inequalities, we obtain

$$\sum_{\text{cyclic}}\frac{a}{b+c}\geq\frac{3}{2} \tag{3}$$

and

$$\sum_{\text{cyclic}}\frac{1}{b+c-a}\geq\frac{9}{\sum\limits_{\text{cyclic}}(b+c-a)}=\frac{9}{a+b+c} \tag{4}$$

Finally, from (2), (3), (4) we get (1) and the proof is finished. Equality holds for $a=b=c$. □

Alternative solution by Ercole Suppa, Teramo, Italy. As in the previous solution it suffices to prove that

$$\sum_{\text{cyclic}}\frac{a}{(b+c)(b+c-a)}\geq\frac{9}{2(a+b+c)} \tag{1}$$

Rewrite (1) in SOS (=sum of squares) form as follows

$$
\begin{aligned}
&\sum_{\text{cyclic}} \frac{a}{(b+c)(b+c-a)} - \frac{9}{2(a+b+c)} \\
=&\sum_{\text{cyclic}} \left[\frac{a}{(b+c)(b+c-a)} - \frac{3}{2(a+b+c)}\right] \\
=&\sum_{\text{cyclic}} \frac{(2a-b-c)(a+3b+3c)}{2(b+c)(b+c-a)(a+b+c)} \\
=&\frac{1}{2(a+b+c)}\left[\sum_{\text{cyclic}} \frac{(a-b)(a+3b+3c)}{(b+c)(b+c-a)} + \sum_{\text{cyclic}} \frac{(a-c)(a+3b+3c)}{(b+c)(b+c-a)}\right] \\
=&\frac{1}{2(a+b+c)}\left[\sum_{\text{cyclic}} \frac{(a-b)(a+3b+3c)}{(b+c)(b+c-a)} + \sum_{\text{cyclic}} \frac{(b-a)(b+3c+3a)}{(c+a)(c+a-b)}\right] \\
=&\frac{1}{2(a+b+c)} \sum_{\text{cyclic}} S_c(a-b)^2
\end{aligned}
$$

where

$$
S_c = \frac{a^2+b^2+7c^2+6ab+8bc++8ca}{(a+c)(b+c)(b+c-a)(a+c-b)}
$$

and S_a, S_b are determined similarly, by cyclic permutation. Since $S_a \geq 0$, $S_b \geq 0$ and $S_c \geq 0$ we have

$$
\sum_{\text{cyclic}} S_c(a-b)^2 \geq 0
$$

and the conclusion follows. $\square$