Problema 11480. Let a, b and c be the lengths of the sides opposite vertices A, B and C, respectively, in a non obtuse triangle. Let h_a, h_b and h_c be the corresponding lengths of the altitudes. Show that

$$\left(\frac{h_a}{a}\right)^2 + \left(\frac{h_b}{b}\right)^2 + \left(\frac{h_c}{c}\right)^2 \geq \frac{9}{4}$$

and determine the cases of equality.

Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria

Solution by Ercole Suppa, Teramo, Italy. Denote by s and Δ the semiperimeter and the area of triangle ABC respectively. Since $h_a = 2\Delta/a$, and simmetrically for b and c, the given inequality is equivalent to

$$\sum_{\text{cyclic}} \left(\frac{2\Delta}{a}\right)^2 \geq \frac{9}{4} \qquad \Longleftrightarrow \qquad \sum_{\text{cyclic}} \frac{1}{a^2} \geq \frac{9}{16\Delta^2} \tag{1}$$

From Heron's formula we have

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} \quad \Leftrightarrow$$
$$16\Delta^2 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4$$

so (1) rewrites as

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{9}{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4} \tag{2}$$

Since $\triangle ABC$ is an non obtuse triangle, we have $\cos A \geq 0$, $\cos B \geq 0$, $\cos C \geq 0$, so

$$x = b^2 + c^2 - a^2 \geq 0, \quad y = a^2 + c^2 - b^2 \geq 0, \quad z = a^2 + b^2 - c^2 \geq 0 \tag{3}$$

From (3) it follows that

$$a^2 = \frac{y+z}{2}, \quad b^2 = \frac{x+z}{2}, \quad c^2 = \frac{x+y}{2}$$

and plugging these in (2), the desired inequality transforms into

$$\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2} \geq \frac{9}{4(xy+yz+zx)} \quad , \quad \forall x,y,z \geq 0$$

which is a known result (Iran TST 1996). Expanding this last one, we obtain

$$\left(\sum_{\text{sym}} x^5y - \sum_{\text{sym}} x^4y^2\right) + 3\left(\sum_{\text{sym}} x^5y - \sum_{\text{sym}} x^3y^3\right) + 2xyz\left(3xyz + \sum_{\text{cyclic}} x^3 - \sum_{\text{sym}} x^2y\right) \geq 0$$

which, according to Muirhead's theorem and Schur's inequality, it's a sum of three nonnegative terms.

Equality holds for $x = y = z$ or $x = y$, $z = 0$ up to permutation, i.e. $a = b = c$ (equilateral triangle) or $a = b$ and $c = a\sqrt{2}$ up to permutations (isosceles right-angled triangle). $\square$