Problema J109. Let a, b, c be positive real numbers. Prove that

$$\frac{(a+b)^2}{c} + \frac{c^2}{a} \geq 4b$$

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The proposed inequality follows from:

$$\frac{(a+b)^2}{c} + \frac{c^2}{a} - 4b =$$

$$= \frac{a^3 + 2a^2b + ab^2 + c^3 - 4abc}{ac} =$$

$$= \frac{a\left[b^2 + 2(a-2c)b + a^2\right] + c^3}{ac} =$$

$$= \frac{a\left[b^2 + 2(a-2c)b + (a-2c)^2 - (a-2c)^2 + a^2\right] + c^3}{ac} =$$

$$= \frac{a\left[(b+a-2c)^2 + 4ac - 4c^2\right] + c^3}{ac} =$$

$$= \frac{a(a+b-2c)^2 + 4a^2c - 4ac^2 + c^3}{ac} =$$

$$= \frac{a(a+b-2c)^2 + c\left(4a^2 - 4ac + c^2\right)}{ac} =$$

$$= \frac{(a+b-2c)^2}{c} + \frac{(2a-c)^2}{a} \geq 0$$

The equality holds if and only if $a + b = 2c$, $c = 2a$ that is if and only if $(a, b, c) = (k, 3k, 2k)$ for some real positive number k. $\square$