Problema J115. Find all positive integers n for which $\sqrt{\sqrt{n}+\sqrt{n+2009}}$ is an integer.

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Let m be an integer such that

$$\sqrt{\sqrt{n}+\sqrt{n+2009}} = m \qquad \Longrightarrow \qquad \sqrt{n}+\sqrt{n+2009} = m^2$$

By setting $x = \sqrt{n+2009}$, $y = \sqrt{n}$ we have

$$\begin{cases} x+y = m^2 \\ \\ x^2 - y^2 = 2009 \end{cases} \Rightarrow \begin{cases} x+y = m^2 \\ \\ x-y = \frac{2009}{m^2} \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2}\left(m^2 + \frac{2009}{m^2}\right) \\ \\ y = \frac{1}{2}\left(m^2 + \frac{2009}{m^2}\right) \end{cases}$$

Therefore

$$4n = 4y^2 = m^4 + \frac{2009^2}{m^4} - 4018 \qquad (*)$$

and this implies that m^4 divides 2009^2.

The divisors of 2009^2 are

$\{1, 7, 41, 49, 287, 343, 1681, 2009, 2401, 11767, 14063, 82369, 98441, 576583, 4036081\}$

and, between them, the only fourth powers are $m = 1$ and $m = 2401 = 7^4$. If $m = 1$ from (*) follows that $n = 1008016$ which not verify the required condition, whereas if $m = 7$ we have $n = 16$ and

$$\sqrt{\sqrt{16}+\sqrt{16+2009}} = 7$$

Thus the only n for which $\sqrt{\sqrt{n}+\sqrt{n+2009}}$ is an integer is $n = 16$. $\square$