Problema J116. A bug is situated in one of the vertices of a cube. Each day it travels to another vertex of a cube. How many six day journeys end at the original vertex?

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First solution by Ercole Suppa, Teramo, Italy

Denote the vertices of the cube with the numbers $1, 2, \ldots, 8$ as shown in Figure 1. Consider the graph $G = (V, E)$ obtained from the vertices and the edges of the cube. A six day journey which starts from 1 and ends at 1 can be represented as a sequence $(x_1, x_2, \cdots, x_7)$ of adjacent vertices, that is, there is an edge from each vertex x_i to the next vertex x_{i+1}.
We call *good* journey a six day journey $(x_1, \cdots, x_7)$ such that $x_1 = x_7 = 1$.

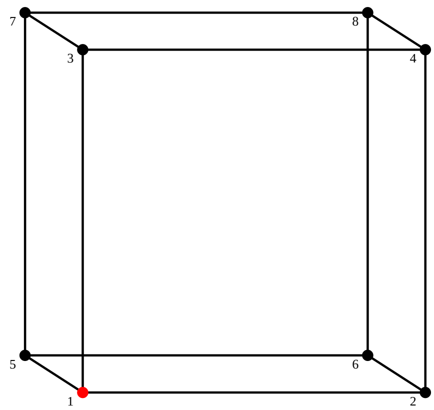

Figure 1

The set Ω of the *good* journeys can be partitioned in four subsets:

- A, the set of the *good* journeys which contains the vertex 8;
- B, C, D, the sets of the *good* journeys which do not contain the vertex 8 and starts with $(1, 2, \cdots)$, $(1, 3, \cdots)$, $(1, 5, \cdots)$ respectively.

Since A, B, C, D are disjoint, by addition rule, we have:

$$\mid \Omega \mid = \mid A \mid + \mid B \mid + \mid C \mid + \mid D \mid$$

Then, in order to solve the problem it is enough to find the number of elements in each of the subsets A, B, C, D.

We see easily that every $c \in A$ must contain 8 in the fourth coordinate, i.e. $c = (1, x, y, 8, u, v, 1)$. Thus, by the product rule, we have:

$$\mid A \mid = 3 \cdot 2 \cdot 3 \cdot 2 = 36$$

Now let us calculate the cardinality of B, C, D. First of all, due to symmetry, we have $\mid B \mid = \mid C \mid = \mid D \mid$. Furthermore observe that there is a bijection between B and the four day journeys which start from 2 and end at 2, 3 or 5. Therefore we can determine the cardinality of B by means of tree enumeration method, as shown in the next three figures:

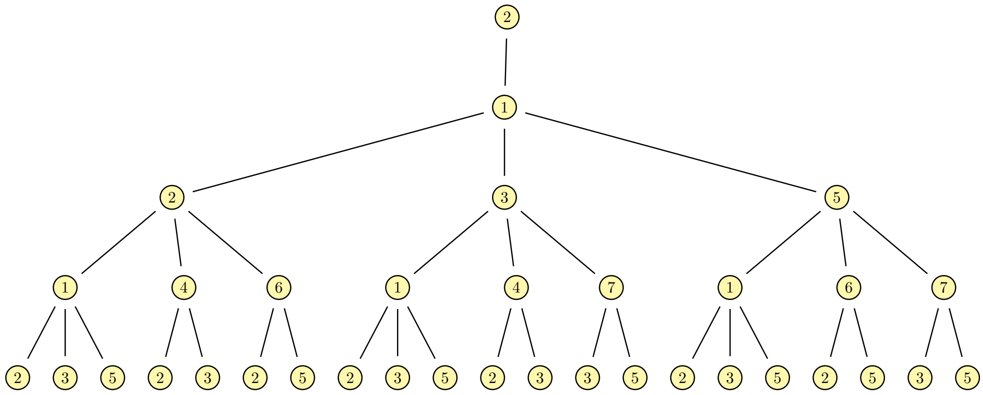

FIGURE 2

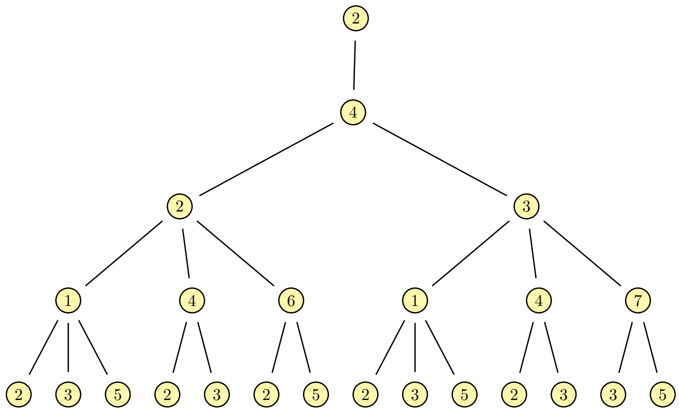

FIGURE 3

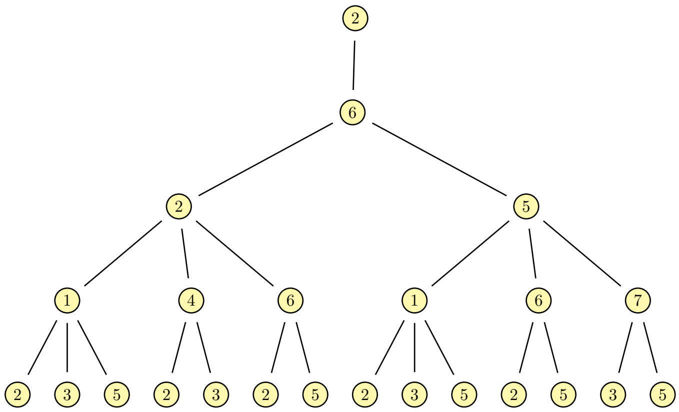

FIGURE 4

A direct counting shows that $\mid B \mid = 49$, so

$$\mid \Omega \mid = 36 + 3 \cdot 49 = 183$$

and we are done. $\square$

Second solution by Ercole Suppa, Teramo, Italy

Denote the vertices of the cube with the numbers $1, 2, \ldots, 8$ as shown in Figure 1 (of first solution). Consider the graph $G = (V, E)$ obtained from the vertices and the edges of the cube and let $A = (a_{ij})$ be its adjacency matrix:

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Writing the (i, j)-entry $a_{ij}^{(2)}$ of the matrix A^2 in the form

$$a_{ij}^{(2)} = \sum_{k=1}^{n} a_{ik} a_{kj}$$

we see that $a_{ij}^{(2)}$ counts the number of intermediate nodes that are connected to both i and j. In other words, $a_{ij}^{(2)}$ gives the number of different walks of length two from i to j. More in general it is not difficult to show that the (i, j)-entry of the matrix A^n is equal to the number of walks of length n that originate at vertex i and terminate at vertex j.

Therefore the number of six day journeys is given by the $(1, 1)$ entry of the matrix A^6:

$$A^6 = \begin{pmatrix} 183 & 0 & 0 & 182 & 0 & 182 & 182 & 0 \\ 0 & 183 & 182 & 0 & 182 & 0 & 0 & 182 \\ 0 & 182 & 183 & 0 & 182 & 0 & 0 & 182 \\ 182 & 0 & 0 & 183 & 0 & 182 & 182 & 0 \\ 0 & 182 & 182 & 0 & 183 & 0 & 0 & 182 \\ 182 & 0 & 0 & 182 & 0 & 183 & 182 & 0 \\ 182 & 0 & 0 & 182 & 0 & 182 & 183 & 0 \\ 0 & 182 & 182 & 0 & 182 & 0 & 0 & 183 \end{pmatrix}$$

i.e. there are 183 six day journeys which start from the vertex 1 and end in vertex 1. $\square$