

Problema J119. Let α, β, γ be angles of a triangle. Prove that

$$\cos^3 \frac{\alpha}{2} \sin \frac{\beta - \gamma}{2} + \cos^3 \frac{\beta}{2} \sin \frac{\gamma - \alpha}{2} + \cos^3 \frac{\gamma}{2} \sin \frac{\alpha - \beta}{2} = 0$$

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Taking into account the product to sum and half-angle identitities

$$\begin{aligned}\cos x \sin y &= \frac{1}{2} [\sin(x+y) - \sin(x-y)] \\ \cos^2 \frac{x}{2} &= \frac{1 + \cos x}{2}\end{aligned}$$

we have

$$\begin{aligned}\sum_{\text{cyclic}} \cos^3 \frac{\alpha}{2} \sin \frac{\beta - \gamma}{2} &= \sum_{\text{cyclic}} \cos^2 \frac{\alpha}{2} \cos \frac{\alpha}{2} \sin \frac{\beta - \gamma}{2} = \\ &= \sum_{\text{cyclic}} \frac{1 + \cos \alpha}{4} \left[\sin \frac{\alpha + \beta - \gamma}{2} - \sin \frac{\alpha - \beta + \gamma}{2} \right] = \\ &= \sum_{\text{cyclic}} \frac{1 + \cos \alpha}{4} \left[\sin \left(\frac{\pi}{2} - \gamma \right) - \sin \left(\frac{\pi}{2} - \beta \right) \right] = \\ &= \frac{1}{4} \sum_{\text{cyclic}} (1 + \cos \alpha) (\cos \gamma - \cos \beta) = \\ &= \frac{1}{4} \sum_{\text{cyclic}} (\cos \gamma - \cos \beta + \cos \alpha \cos \gamma - \cos \alpha \cos \beta) = 0\end{aligned}$$

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