

**Problema J120.** Let  $a, b, c$  be positive real numbers. Prove that

$$\frac{ab}{3a+4b+2c} + \frac{bc}{3b+4c+2a} + \frac{ca}{3c+4a+2b} \leq \frac{a+b+c}{9}$$

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Clearing the denominators the inequality can be rewritten as

$$12a^4 + 16a^3b + 9a^2b^2 + 26ab^3 + 12b^4 + 26a^3c - 63a^2bc - 63ab^2c + 16b^3c + 9a^2c^2 - 63abc^2 + 9b^2c^2 + 16ac^3 + 26bc^3 + 12c^4 \geq 0$$

or in the following cyclic form:

$$\sum_{\text{cyclic}} (12a^4 + 16a^3b + 9a^2b^2 + 26ab^3 - 63a^2bc) \geq 0 \iff 12 \sum_{\text{cyclic}} (a^4 - a^2bc) + 16 \sum_{\text{cyclic}} (a^3b - a^2bc) + 9 \sum_{\text{cyclic}} (a^2b^2 - a^2bc) + 26 \sum_{\text{cyclic}} (ab^3 - a^2bc) \geq 0$$

Therefore, taking into account the following identities

$$\sum_{\text{cyclic}} (a^4 - a^2bc) = \sum_{\text{cyclic}} \frac{1}{12} (5b^2 + 6bc + 7c^2 + 8ab + 4ac) (b - c)^2 \quad (1)$$

$$\sum_{\text{cyclic}} (a^3b - a^2bc) = \sum_{\text{cyclic}} \frac{1}{3} (-b^2 + c^2 + 2ab + ac) (b - c)^2 \quad (2)$$

$$\sum_{\text{cyclic}} (a^2b^2 - a^2bc) = \sum_{\text{cyclic}} \frac{1}{12} (-b^2 - 6bc + c^2 + 8ab + 4ac) (b - c)^2 \quad (3)$$

$$\sum_{\text{cyclic}} (ab^3 - a^2bc) = \sum_{\text{cyclic}} \frac{1}{6} (b^2 - c^2 + 4ab + 2ac) (b - c)^2 \quad (4)$$

the given inequality is equivalent to

$$S_a(b - c)^2 + S_b(a - c)^2 + S_c(a - b)^2 \geq 0 \quad (5)$$

where the coefficients  $S_a, S_b, S_c$  are:

$$\begin{aligned} S_a &= 168ab + 13b^2 + 84ac + 6bc + 35c^2 \\ S_b &= 35a^2 + 84ab + 6ac + 168bc + 13c^2 \\ S_c &= 13a^2 + 6ab + 35b^2 + 168ac + 84bc \end{aligned}$$

The inequality (5) is true because  $S_a, S_b, S_c$  are positive real numbers.  
Equality holds for  $a = b = c$ .  $\square$