Problema J122. Let $ABCD$ be a quadrilateral inscribed in a circle and circumscribed about a circle such that the points of tangency form a quadrilateral $A_1B_1C_1D_1$. Prove that $A_1C_1 \perp B_1D_1$:

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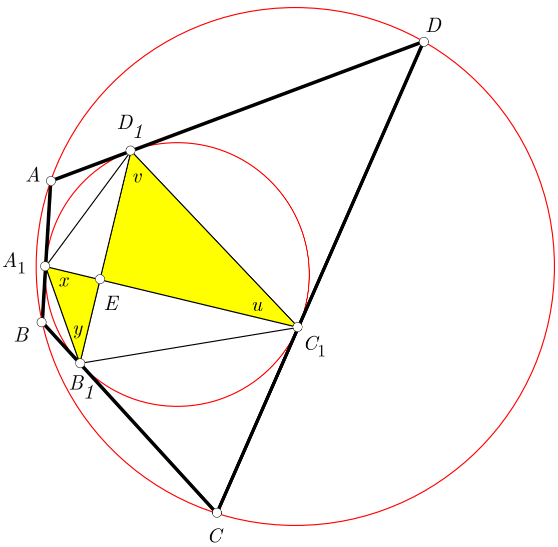
Let us denote $E = A_1C_1 \cap B_1D_1$, $x = \angle B_1A_1E$, $y = \angle A_1B_1E$, $u = \angle EC_1D_1$, $v = \angle ED_1C_1$, as shown in figure.



From the angle identities $x = \angle CB_1C_1$ and $v = \angle B_1C_1C$ we have:

$$x + v = \angle CB_1C_1 + \angle B_1C_1C = 180^\circ - \angle C \tag{1}$$

Similarly we obtain

$$y + u = \angle AD_1A_1 + \angle AA_1D_1 = 180^\circ - \angle A \tag{2}$$

Adding (1), (2) and using the fact that $ABCD$ is circumscribed about a circle we get

$$x + y + u + v = 360^\circ - (\angle A + \angle C) = 180^\circ \tag{3}$$

Since the sum of the angles of triangles $\triangle D_1EC_1$ and $\triangle A_1B_1E$ is 360°, from (3) it follows that

$$\angle A_1EB_1 + \angle C_1ED_1 = 180^\circ \tag{4}$$

Therefore, taking into account that $\angle A_1EB_1 = \angle C_1ED_1$, we have

$$\angle A_1EB_1 = \angle C_1ED_1 = 90^\circ \qquad \Longleftrightarrow \qquad A_1C_1 \perp B_1D_1$$

and the proof is complete. □