Problema J123. Solve in prime numbers the equation: $x^y + y^x = z$.

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Since z is a prime number x and y must have different parity. Without loss of generality we can assume that x is an odd prime and $y = 2$. The given equation rewrites as

$$x^2 + 2^x = z \qquad , \qquad x, z \text{ prime numbers} \tag{1}$$

Taking into account that $2^x \equiv -1 \pmod 3$, it follows that $x \equiv 0 \pmod 3$, because if $x \equiv 1$ or $x \equiv 2 \pmod 3$ we should have $x^2 \equiv 1 \pmod 3$ so, by (1) :

$$z = x^2 + 2^x \equiv 0 \pmod 3 \quad \Rightarrow \quad z = 3 \text{ (since } z \text{ is prime)} \quad \Rightarrow \quad x = 1$$

and this is impossible since x is a prime number.

Now $x \equiv 0 \pmod 3$ yields $x = 3$, since x is a prime number. Therefore the only solutions are $(3, 2, 17)$ and $(2, 3, 17)$, by simmetry. $\square$