Problema J124. Let a and b be integers such that $|b-a|$ is an odd prime. Prove thet $P(x) = (x-a)(x-b) - p$ is irreducible in $\mathbb{Z}[X]$ for any prime p.

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Assume by contradiction that $P(x)$ is reducible. Then there exists $c \in \mathbb{Z}$ such that $P(c) = 0$, i.e.

$$p = (c-a)(c-b)$$

Since p is prime, one of the factors $c-a$ or $c-b$ must be equal to $\pm p$. Without loss of generality we can suppose that

$$c - a = \pm p \qquad , \qquad c - b = \pm 1 \tag{1}$$

From (1) it follows that

$$|b-a| = |1-p| \tag{2}$$

so $p = 2$, because $|b-a|$ is odd and p is prime. Then, from (2), we get $|b-a| = 1$ reaching a contradiction because $|b-a|$ is prime. The proof is finished. $\square$