Problema J125. Let ABC be an isosceles triangle with $\angle A = 100^\circ$. Denote by BL the angle bisector of angle $\angle ABC$. Prove that $AL + BL = BC$.

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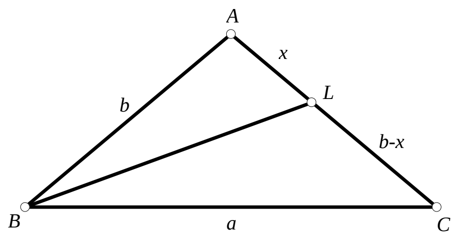
Let us denote $BC = a$, $AB = AC = b$, $AL = x$ as in figure.



The internal bisector theorem yields:

$$x : (b - x) = b : a \qquad \Rightarrow \qquad AL = x = \frac{b^2}{a+b} \tag{1}$$

The length of the angle bisector BL is given by

$$BL = \frac{2ab}{a+b}\cos 20^\circ \tag{2}$$

In order to complete the proof we need the following identity

$$\cos 20^\circ = \cos 40^\circ + \cos 80^\circ \tag{3}$$

which is true because

$$\cos 40^\circ + \cos 80^\circ = 2\cos\frac{40^\circ + 80^\circ}{2}\cos\frac{40^\circ - 80^\circ}{2} = 2 \cdot \frac{1}{2} \cdot \cos(-20^\circ) = \cos 20^\circ$$

From (1), (2), (3), taking into account that $a = 2b\cos 40^\circ$, we have

$$\begin{aligned}
AL + BL &= \frac{b^2}{a+b} + \frac{2ab}{a+b}\cos 20^\circ = \frac{b^2 + 2ab\cos 20^\circ}{a+b} = \\
&= \frac{b^2 + 4b^2\cos 20^\circ\cos 40^\circ}{2b\cos 40^\circ + b} = \frac{1 + 4\cos 20^\circ\cos 40^\circ}{2\cos 40^\circ + 1}b = \\
&= \frac{1 + 4\cdot\frac{1}{2}\left(\cos 60^\circ + \cos 20^\circ\right)}{2\cos 40^\circ + 1}b = \frac{2 + 2\cos 20^\circ}{2\cos 40^\circ + 1}b = \\
&= \frac{2 + 2\left(\cos 40^\circ + \cos 80^\circ\right)}{2\cos 40^\circ + 1}b = \left(1 + \frac{2\cos 80^\circ + 1}{2\cos 40^\circ + 1}\right)b = \\
&= \left(1 + \frac{2\left(2\cos^2 40^\circ - 1\right) + 1}{2\cos 40^\circ + 1}\right)b = \left(1 + \frac{4\cos^2 40^\circ - 1}{2\cos 40^\circ + 1}\right)b = \\
&= \left(1 + 2\cos 40^\circ - 1\right)b = 2b\cos 40^\circ = a = BC
\end{aligned}$$

and the proof is complete. $\square$