Problema J126. Let a, b, c be positive real numbers. Prove that

$$3\left(a^2b^2+b^2c^2+c^2a^2\right)\left(a^2+b^2+c^2\right)\geq\left(a^2+ab+b^2\right)\left(b^2+bc+c^2\right)\left(c^2+ca+a^2\right)$$

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Due to simmetry we may suppose $a \leq b \leq c$, and put $b = a+u$, $c = a+u+v$ where $u, v \geq 0$. By a tedious calculation it can be readily checked that the given inequality rewrites as

$$\begin{aligned}
&6a^4u^2+18a^3u^3+21a^2u^4+12au^5+3u^6+6a^4uv+27a^3u^2v+\\
&+42a^2u^3v+30au^4v+9u^5v+6a^4v^2+21a^3uv^2+36a^2u^2v^2+\\
&+30au^3v^2+11u^4v^2+6a^3v^3+15a^2uv^3+15au^2v^3+7u^3v^3+\\
&+3a^2v^4+3auv^4+2u^2v^4\geq 0
\end{aligned}$$

Since $a, u, v \geq 0$ the left hand side is non negative, and the inequality is proved. Equality holds for $u = v = 0$, i.e. for $a = b = c$. $\square$