Problema J130. Consider a triangle ABC. Let D the orthogonal projection of A onto BC and let E and F be points on lines AB and AC respectively such that $\angle ADE = \angle ADF$. Prove that the lines AD, BF, and CE are concurrent.

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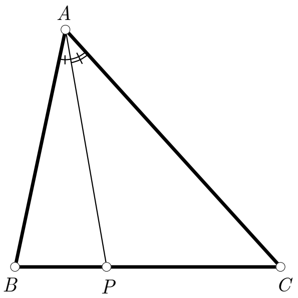
We will prove a lemma first:

LEMMA. If P is a point on the side BC of a triangle $\triangle ABC$ we have

$$\frac{PB}{PC} = \frac{AB}{AC} \cdot \frac{\sin \angle PAB}{\sin \angle PAC}$$

Proof.



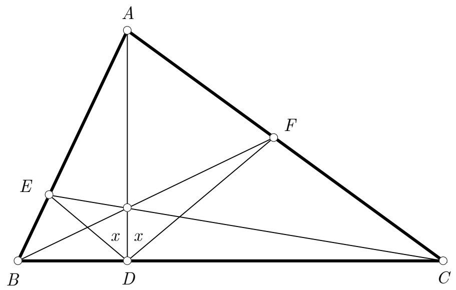
In triangles $\triangle PAB$ and $\triangle PAC$, the law of sines gives

$$\frac{PB}{\sin \angle PAB} = \frac{AB}{\sin \angle APB}$$

$$\frac{PC}{\sin \angle PAC} = \frac{AC}{\sin(180^\circ - \angle APB)} = \frac{AC}{\sin \angle APB}$$

Dividing the above relations we get the desired result. ■

Coming back to the problem, let us denote $x = \angle ADE = \angle ADF$, as shown in figure.



By the Lemma we have

$$\frac{AE}{EB} = \frac{AD}{BD} \cdot \frac{\sin x}{\sin(90^\circ - x)}$$

$$\frac{CF}{FA} = \frac{DC}{AD} \cdot \frac{\sin(90^\circ - x)}{\sin x}$$

Therefore

$$\frac{AE}{EB} \cdot \frac{BD}{DC} \cdot \frac{CF}{FA} = \frac{AD}{BD} \cdot \frac{\sin x}{\sin(90^\circ - x)} \cdot \frac{BD}{DC} \cdot \frac{DC}{AD} \cdot \frac{\sin(90^\circ - x)}{\sin x} = 1$$

so, by Ceva's theorem the lines AD, BF, and CE are concurrent and we are done. $\square$