

**Problema J131.** Let  $P$  be a point inside a triangle  $ABC$  and let  $d_a, d_b, d_c$  be the distances from point  $P$  to the triangle's sides. Prove that

$$d_a h_a^2 + d_b h_b^2 + d_c h_c^2 \geq (d_a + d_b + d_c)^3$$

where  $h_a, h_b, h_c$  are the altitudes of the triangle.

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By denoting with  $\Delta$  be the area of  $\triangle ABC$ , we have

$$\frac{d_a}{h_a} + \frac{d_b}{h_b} + \frac{d_c}{h_c} = \frac{d_a}{\frac{2\Delta}{a}} + \frac{d_a}{\frac{2\Delta}{b}} + \frac{d_a}{\frac{2\Delta}{c}} = \frac{ad_a + bd_b + cd_c}{2\Delta} = 1 \quad (1)$$

By using the Hölder's inequality

$$\sum_{i=1}^n x_i y_i \leq \left( \sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^n y_i^q \right)^{\frac{1}{q}}$$

with  $p = 3, q = \frac{3}{2}$  and

$$x_1 = (d_a h_a^2)^{\frac{1}{3}}, \quad x_2 = (d_b h_b^2)^{\frac{1}{3}}, \quad x_3 = (d_c h_c^2)^{\frac{1}{3}}$$

$$y_1 = \left( \frac{d_a}{h_a} \right)^{\frac{2}{3}}, \quad y_2 = \left( \frac{d_b}{h_b} \right)^{\frac{2}{3}}, \quad y_3 = \left( \frac{d_c}{h_c} \right)^{\frac{2}{3}}$$

we get

$$\begin{aligned} d_a + d_b + d_c &= \sum (d_a h_a^2)^{\frac{1}{3}} \left( \frac{d_a}{h_a} \right)^{\frac{2}{3}} \leq \left( \sum d_a h_a^2 \right)^{\frac{1}{3}} \left( \sum \frac{d_a}{h_a} \right)^{\frac{2}{3}} = \\ &= (d_a h_a^2 + d_b h_b^2 + d_c h_c^2)^{\frac{1}{3}} \left( \frac{d_a}{h_a} + \frac{d_b}{h_b} + \frac{d_c}{h_c} \right)^{\frac{2}{3}} \end{aligned} \quad (2)$$

From (1) and (2) it follows that

$$\begin{aligned} d_a + d_b + d_c &\leq (d_a h_a^2 + d_b h_b^2 + d_c h_c^2)^{\frac{1}{3}} \Rightarrow \\ (d_a + d_b + d_c)^3 &\leq d_a h_a^2 + d_b h_b^2 + d_c h_c^2 \end{aligned}$$

and we are done.  $\square$