Problema J134. How many positive integers n less than 2009 are divisible by $\lfloor \sqrt[3]{n} \rfloor$?

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Let us say a positive integer n is *nice* if it is divisible by $\lfloor \sqrt[3]{n} \rfloor$. Denote by $N(X)$ the number of nice numbers contained in a set X.

Since $11^3 < 2009 < 12^3$, in order to count how many *nice* number are less than 2009, consider the following partition of $X = \{1, 2, \ldots, 2009\}$:

$$\begin{aligned} X_k &= \left\{k^3, k^3+1, \ldots, (k+1)^3 - 1\right\}, \qquad k = 1, 2, \ldots, 11 \\ X_{12} &= \left\{12^3, 1729, 2009\right\} \end{aligned}$$

We have $[\sqrt[3]{x}] = k$ for every $x \in X_k$ so

$$\begin{aligned} N\left(X_k\right) &= \left[\frac{(k+1)^3 - 1}{k}\right] - \left[\frac{k^3 - 1}{k}\right] = \\ &= \left[(k+1)^2 + k + 2\right] - \left[k^2 - \frac{1}{k}\right] = \\ &= k^2 + 3k + 3 - \left(k^2 - 1\right) = 3k + 4 \end{aligned}$$

Therefore

$$\begin{aligned} N(X) &= \sum_{k=1}^{12} N\left(X_k\right) = \sum_{k=1}^{11} (3k+4) + N\left(X_{12}\right) = \\ &= 3\sum_{k=1}^{11} k + 44 + \left[\frac{2009}{12}\right] - \left[\frac{1727}{12}\right] = \\ &= 198 + 44 + 167 - 143 = 266 \end{aligned}$$

and we are done. $\square$