Problema J135. Find all n for which the number of diagonals of a convex n-gon is a perfect square.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Ercole Suppa, Teramo, Italy

The number of diagonals of a convex n-gon is given by

$$\binom{n}{2} - n = \frac{n(n-1)}{2} - n = \frac{n^3 - n}{2}$$

so it is enough to solve the following diophantine equation

$$n^2 - 3n = 2m^2, \qquad n, m \in \mathbb{N} \tag{1}$$

From (1) follows that

$$n^2 - 3n \equiv 2m^2 \pmod 3 \qquad \Rightarrow \qquad n \equiv 0, m \equiv 0 \pmod 3$$

Thus, setting $n = 3a$ and $m = 3y$ ($a, y \in \mathbb{N}$), the equation (1) is equivalent to

$$\begin{aligned}
9a^2 - 9a &= 18y^2 && \Leftrightarrow \\
a^2 - a &= 2y^2 && \Leftrightarrow \\
4a^2 - 4a + 1 &= 8y^2 + 1 && \Leftrightarrow \\
(2a-1)^2 - 8y^2 &= 1 && \Leftrightarrow \\
x^2 - 8y^2 &= 1 && && (2)
\end{aligned}$$

where we have put $x = 2a - 1$.

The equation (2) is a Pell's equation $x^2 - Dy^2 = 1$ with fundamental solution $x_1 = 3$, $y_1 = 1$, so all positive solutions are of the form x_k, y_k, where

$$x_k + y_k\sqrt{8} = \left(x_1 + y_1\sqrt{D}\right)^k$$

The solutions x_k, y_k can be computed from the formulas

$$\begin{cases} x_k = \frac{1}{2}\left[\left(x_1 + y_1\sqrt{D}\right)^k + \left(x_1 - y_1\sqrt{D}\right)^k\right] \\ y_k = \frac{1}{2\sqrt{D}}\left[\left(x_1 + y_1\sqrt{D}\right)^k + \left(x_1 - y_1\sqrt{D}\right)^k\right] \end{cases}$$

or from the recursive formulas

$$\begin{cases} x_{k+1} = x_1 x_k + D y_1 y_k \\ y_{k+1} = x_1 y_k + x_k y_1 \end{cases} \quad \Longleftrightarrow \quad \begin{cases} x_{k+1} = 3x_k + 8y_k \\ y_{k+1} = 3y_k + x_k \end{cases} \tag{3}$$

From (3), by a simple calculation, we get

$$x_{k+2} = 6 \cdot x_{k+1} - x_k \qquad , \qquad x_1 = 3, x_2 = 17 \tag{4}$$

Since $n = 3a = \frac{3}{2}(x+1)$ the recurrence (4) yields

$$\frac{3}{2}\left(x_{k+2}+1\right) = 6 \cdot \frac{3}{2}\left(x_{k+1}+1\right) - \frac{3}{2}\left(x_k+1\right) + 6 \qquad \Longleftrightarrow$$

$$\boxed{n_{k+2} = 6n_{k+1} - n_k + 6 \qquad , \qquad n_1 = 6, n_2 = 27} \tag{5}$$

By means of MATHEMATICA we have listed the first 10 solutions given by (5):

$$6, 27, 150, 867, 5046, 29403, 171366, 998787, 5821350, 33929307$$

The proof is finished. $\square$