

**Problema J136.** Let  $a, b, c$  be the sides,  $m_a, m_b, m_c$  the medians,  $h_a, h_b, h_c$  the altitudes, and  $l_a, l_b, l_c$  the angle bisectors of a triangle  $ABC$ . Prove that the diameter of the circumcircle of triangle  $ABC$  is equal to

$$\frac{l_a^2}{h_a} \sqrt{\frac{m_a^2 - h_a^2}{l_a^2 - h_a^2}}$$

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In order to prove the required identity we will use the well-known formulas

$$m_a = \frac{1}{2} \sqrt{2b^2 + 2c^2 - a^2}, \quad l_a = \frac{2bc}{b+c} \sqrt{\frac{s(s-a)}{bc}}, \quad h_a = \frac{2\Delta}{a} \quad (1)$$

where  $s = \frac{a+b+c}{2}$  and  $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$  is the area of  $\triangle ABC$ .

From (1) it follows that

$$\frac{l_a^2}{h_a} = \frac{4b^2c^2s(s-a)}{bc(b+c)^2} \cdot \frac{a}{2\Delta} = \frac{2abc\Delta}{(b+c)^2(s-b)(s-c)} \quad (2)$$

$$\begin{aligned} m_a^2 - h_a^2 &= \frac{1}{4} (2b^2 + 2c^2 - a^2) - \frac{4s(s-a)(s-b)(s-c)}{a^2} = \\ &= \frac{a^2 (2b^2 + 2c^2 - a^2) - [(b+c)^2 - a^2] [a^2 - (b-c)^2]}{4a^2} = \\ &= \frac{(b^2 - c^2)^2}{4a^2} \end{aligned} \quad (3)$$

$$\begin{aligned} l_a^2 - h_a^2 &= \frac{4bc(s-a)}{(b+c)^2} - \frac{4\Delta^2}{a^2} = \\ &= s(s-a) \cdot \frac{4a^2bc - 4(b+c)^2(s-b)(s-c)}{a^2(b+c)^2} = \\ &= s(s-a) \cdot \frac{4a^2bc - (b+c)^2(a+c-b)(a+b-c)}{a^2(b+c)^2} = \\ &= s(s-a) \cdot \frac{4a^2bc - a^2(b+c)^2 + (b^2 - c^2)^2}{a^2(b+c)^2} = \\ &= s(s-a) \cdot \frac{(b^2 - c^2)^2 - a^2(b-c)^2}{a^2(b+c)^2} = \\ &= s(s-a)(b-c)^2 \cdot \frac{(b+c)^2 - a^2}{a^2(b+c)^2} = \\ &= \frac{4s^2(s-a)^2(b-c)^2}{a^2(b+c)^2} \end{aligned} \quad (4)$$

Now, by using (2), (3), (4), we get

$$\begin{aligned}
\frac{l_a^2}{h_a} \sqrt{\frac{m_a^2 - h_a^2}{l_a^2 - h_a^2}} &= \frac{2abc\Delta}{(b+c)^2(s-b)(s-c)} \sqrt{\frac{(b^2 - c^2)^2}{4a^2} \cdot \frac{a^2(b+c)^2}{4s^2(s-a)^2(b-c)^2}} = \\
&= \frac{2abc\Delta}{(b+c)^2(s-b)(s-c)} \sqrt{\frac{(b+c)^4}{16s^2(s-a)^2}} = \\
&= \frac{abc\Delta}{s(s-a)(s-b)(s-c)} = \frac{abc}{2\Delta} = 2R
\end{aligned}$$

and the proof is complete.  $\square$