Problema J137. Let ABC be a triangle and let tangents to the circumcircle at A, B, C intersect BC, AC, AB at points A_1, B_1, C_1, respectively. Prove that

$$\frac{1}{AA_1}+\frac{1}{BB_1}+\frac{1}{CC_1}=2\max\left(\frac{1}{AA_1},\frac{1}{BB_1},\frac{1}{CC_1}\right)$$

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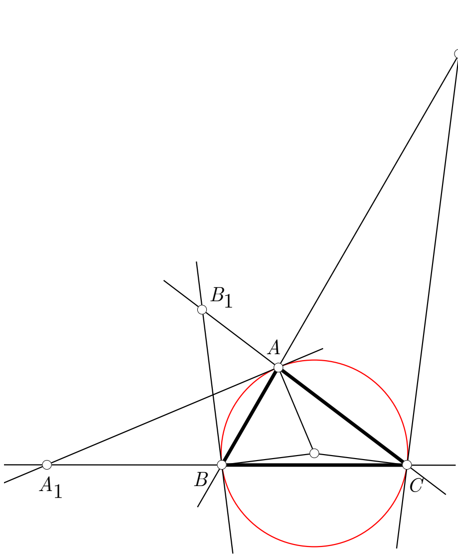
Let us denote $a = BC$, $b = AC$, $c = AB$, $\alpha = \angle BAC$, $\beta = \angle ABC$, $\gamma = \angle ACB$ and suppose without loss of generality that $a \geq b \geq c$ (so $\alpha \geq \beta \geq \gamma$).



Since $\angle A_1AB = \gamma$ we have $\angle A_1AC = \alpha+\gamma$. Therefore the sinus theorem yields

$$\frac{AA_1}{\sin\gamma}=\frac{AC}{\sin(180^\circ-\alpha-2\gamma)} \quad\Rightarrow\quad AA_1=\frac{b\sin\gamma}{\sin(\beta-\gamma)} \quad\Rightarrow$$

$$\frac{1}{AA_1}=\frac{\sin\beta\cos\gamma-\cos\beta\sin\gamma}{b\sin\gamma}=\frac{1}{2R}\cot\gamma-\frac{\cos\beta}{b}=\frac{1}{2R}(\cot\gamma-\cot\beta) \tag{1}$$

Similarly we obtain

$$\frac{1}{BB_1}=\frac{1}{2R}(\cot\gamma-\cot\alpha) \tag{2}$$

and

$$\frac{1}{CC_1}=\frac{1}{2R}(\cot\beta-\cot\alpha) \tag{3}$$

Adding (1), (2), (3) we get

$$\frac{1}{AA_1}+\frac{1}{BB_1}+\frac{1}{CC_1}=\frac{1}{R}(\cot\gamma-\cot\alpha) \tag{4}$$

In order to complete the proof we must show that

$$\max\left(\frac{1}{AA_1}, \frac{1}{BB_1}, \frac{1}{CC_1}\right) = \frac{1}{BB_1} = \frac{1}{2R}\left(\cot\gamma - \cot\alpha\right) \tag{5}$$

Now, if $\triangle ABC$ is an acute-angled triangle, $\cot\gamma \geq \cot\beta \geq \cot\alpha > 0$ and this implies that

$$\cot\gamma - \cot\alpha \geq \cot\gamma - \cot\beta \qquad \Longleftrightarrow \qquad \frac{1}{BB_1} \geq \frac{1}{AA_1} \tag{6}$$

$$\cot\gamma - \cot\alpha \geq \cot\beta - \cot\alpha \qquad \Longleftrightarrow \qquad \frac{1}{BB_1} \geq \frac{1}{CC_1} \tag{7}$$

whereas if $\triangle ABC$ is an obtuse-angled or a right triangle we have $\cot\gamma \geq \cot\beta$, $\cot\alpha \leq 0$ and the relations (6) and (7) are verified anyway. The proof is complete. □