Problema J138. Let a, b, c be positive real numbers. Prove that

$$\frac{a^3}{b^2+c^2}+\frac{b^3}{c^2+a^2}+\frac{c^3}{a^2+b^2}\geq\frac{a+b+c}{2}$$

Proposed by Mircea Becheanu, University of Bucarest, Romania

Solution by Ercole Suppa, Teramo, Italy

Because of the symmetry we may assume that $a\leq b\leq c$. Thus we have $a^3\leq b^3\leq c^3$ and

$$\frac{1}{b^2+c^2}\leq\frac{1}{c^2+a^2}\leq\frac{1}{a^2+b^2}$$

so the rearrangement inequality yields

$$\frac{a^3}{b^2+c^2}+\frac{b^3}{c^2+a^2}+\frac{c^3}{a^2+b^2}\geq\frac{a^3}{a^2+b^2}+\frac{b^3}{b^2+c^2}+\frac{c^3}{c^2+a^2}\tag{1}$$

Therefore, according to (1), it suffices to prove that

$$\frac{a^3}{a^2+b^2}+\frac{b^3}{b^2+c^2}+\frac{c^3}{c^2+a^2}\geq\frac{a+b+c}{2}\tag{2}$$

Now, from AM-GM inequality, we have the following estimation

$$\frac{a^3}{a^2+b^2}=\frac{a^3+ab^2-ab^2}{a^2+b^2}=a-\frac{ab^2}{a^2+b^2}\geq a-\frac{ab^2}{2ab}=a-\frac{b}{2}\tag{3}$$

Adding up (3) and similar cyclic results we get (2), so the desired inequality is proved. $\square$