Problema J139. Let $a_0 = a_1 = 1$ and

$$a_{n+1} = \frac{a_n^2}{a_n + a_{n-1}}$$

for $n \geq 1$. Find a_n in closed form.

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Clearly, all the terms of the sequence are positive integers and for each $n \geq 1$ we have

$$a_n^2 = a_{n+1}a_n + a_{n+1}a_{n-1}$$

which is equivalent to

$$\frac{a_n}{a_{n+1}} = 1 + \frac{a_{n-1}}{a_n}$$

Therefore, the sequence $b_n = \frac{a_n}{a_{n+1}}$ satisfies the recursive relation

$$b_n = 1 + b_{n-1} \quad \text{for all } n \geq 1$$

i.e. b_n is an arithmetic progression with $b_0 = 1$ and common difference $d = 1$.

This implies that $b_n = n$, which is equivalent to

$$a_{n+1} = \frac{1}{n} \cdot a_n \quad \text{for all } n \geq 1$$

Now, a simple inductive argument shows that

$$a_n = \frac{1}{n!}$$

and the result is established. $\square$