

Problema J140. Let n be a positive integer. Find all real numbers x such that

$$\lfloor x \rfloor + \lfloor 2x \rfloor + \cdots + \lfloor nx \rfloor = \frac{n(n+1)}{2}$$

Proposed by Mihai Piticari, "Dragos-Voda" National College, Romania

Solution by Ercole Suppa, Teramo, Italy

For each positive integer n define the function

$$f_n(x) = \lfloor x \rfloor + \lfloor 2x \rfloor + \cdots + \lfloor nx \rfloor$$

Clearly, $f_n(x)$ is nondecreasing. Observe that:

- if $x < 1$ then

$$f_n(x) < 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

- if $x = 1$ then

$$f_n(1) = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

- if $1 \leq x < 1 + \frac{1}{n}$ then $k \leq kx < k + \frac{k}{n}$ for all $k = 2, 3, \dots, n$, so

$$f_n(x) = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

- if $x = 1 + \frac{1}{n}$ then

$$\begin{aligned} f_n\left(1 + \frac{1}{n}\right) &= \left\lfloor 1 + \frac{1}{n} \right\rfloor + \left\lfloor 2 + \frac{2}{n} \right\rfloor + \cdots + \left\lfloor n - 1 + \frac{n-1}{n} \right\rfloor + \left\lfloor n + \frac{n}{n} \right\rfloor = \\ &= 1 + 2 + \cdots + (n-1) + (n+1) = \\ &= \frac{n(n+1)}{2} + 1 \end{aligned}$$

Therefore, our equation is satisfied for all $x \in \mathbb{R}$ such that $1 \leq x < 1 + \frac{1}{n}$. \square