Problema J141. Let a, b, c be the side lengths of a triangle. Prove that

$$0 \le \frac{a-b}{b+c} + \frac{b-c}{c+a} + \frac{c-a}{a+b} < 1$$

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The given inequality is equivalent to

$$\begin{aligned} 0 \le \frac{a+c}{b+c} + \frac{b+a}{c+a} + \frac{c+b}{a+b} - 3 < 1 \qquad & \Leftrightarrow \\ 3 \le \frac{a+c}{b+c} + \frac{b+a}{c+a} + \frac{c+b}{a+b} < 4 \qquad & \end{aligned} \tag{1}$$

The left-hand side of (1) follows from **AM-GM** inequality, since

$$\frac{a+c}{b+c} + \frac{b+a}{c+a} + \frac{c+b}{a+b} \ge 3\sqrt[3]{\frac{a+c}{b+c} \cdot \frac{b+a}{c+a} \cdot \frac{c+b}{a+b}} = 3\sqrt[3]{1} = 3$$

For the right-hand side of (1) note that the triangle inequality yields

$$b+c > a \qquad \Rightarrow \qquad b+c > \frac{a+b+c}{2} \tag{2}$$

Adding (2) and similar cyclic results we get

$$\frac{a+c}{b+c} + \frac{b+a}{c+a} + \frac{c+b}{a+b} < \frac{a+c}{\frac{a+b+c}{2}} + \frac{b+a}{\frac{a+b+c}{2}} + \frac{c+b}{\frac{a+b+c}{2}} = \frac{4(a+b+c)}{a+b+c} = 4$$

and the desired result is proved. $\square$