Problema J144. Let ABC be a triangle with $a > b > c$. Denote by O and H its circumcenter and orthocenter, respectively. Prove that

$$\sin\angle AHO + \sin\angle BHO + \sin\angle CHO \leq \frac{(a-c)(a+c)^3}{4abc \cdot OH}$$

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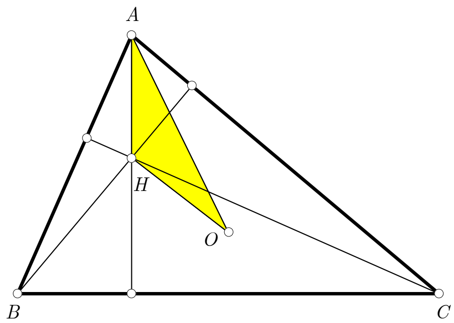
Let R, A, B, C, a, b, c be the circumradius, the angles and the side lengths of the triangle ABC, respectively.



Clearly we have

$$\angle HAO = \angle HAC - \angle OAC = (90^\circ - C) - (90^\circ - B) = B - C \tag{1}$$

The sine law in triangle AHO yields

$$\frac{AO}{\sin\angle AHO} = \frac{OH}{\sin\angle HAO} \tag{2}$$

From (1) and (2), taking into account the law of sines and the law of cosines in triangle ABC, it follows that

$$\begin{aligned}
\sin\angle AHO &= \frac{R}{OH} \cdot \sin(B-C) = \\
&= \frac{R}{OH} \cdot (\sin B\cos C - \cos B \sin C) = \\
&= \frac{1}{OH} \cdot \left(\frac{b}{2}\cos C - \frac{c}{2}\cos B\right) = \\
&= \frac{1}{OH} \cdot \left(b \cdot \frac{a^2+b^2-c^2}{4ab} - c \cdot \frac{a^2+c^2-b^2}{4ac}\right) = \\
&= \frac{1}{OH} \cdot \frac{b^2-c^2}{2a}
\end{aligned}$$

Building up two similar equalities and adding up all of them, we get

$$
\begin{aligned}
&\sin\angle AHO + \sin\angle BHO + \sin\angle CHO = \\
&= \frac{1}{OH}\cdot\left(\frac{b^2-c^2}{2a}+\frac{a^2-c^2}{2b}+\frac{a^2-b^2}{2c}\right) = \\
&= \frac{1}{2abc\cdot OH}\left[bc\left(b^2-c^2\right)+ac\left(a^2-c^2\right)+ab\left(a^2-b^2\right)\right] = \\
&= \frac{1}{2abc\cdot OH}(a+b)(b+c)(a-c)(a-b+c) = \\
&= \frac{1}{4abc\cdot OH}(a+b)(b+c)(a-c)(2a-2b+2c)
\end{aligned}
$$

Then, according to the above relation, the given inequality can be rewritten in the form

$$(a+b)(b+c)(2a-2b+2c) \le (a+c)^3$$

which is true because of **AM-GM** inequality. □