

**Problema J146.** Let  $A_1A_2A_3A_4A_5$  be a convex pentagon and let  $X \in A_1A_2$ ,  $Y \in A_2A_3$ ,  $Z \in A_3A_4$ ,  $U \in A_4A_5$ ,  $V \in A_5A_1$  be points such that  $A_1Z$ ,  $A_2U$ ,  $A_3V$ ,  $A_4X$ ,  $A_5Y$  intersect at  $P$ . Prove that

$$\frac{A_1X}{A_2X} \cdot \frac{A_2Y}{A_3Y} \cdot \frac{A_3Z}{A_4Z} \cdot \frac{A_4U}{A_5U} \cdot \frac{A_5V}{A_1V} = 1$$

*Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA*

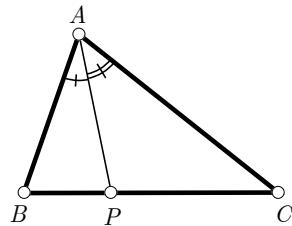
*Solution by Ercole Suppa, Teramo, Italy*

We will use the following

LEMMA. If  $P$  is a point on the side  $BC$  of a triangle  $\triangle ABC$  we have

$$\frac{PB}{PC} = \frac{AB}{AC} \cdot \frac{\sin \angle PAB}{\sin \angle PAC}$$

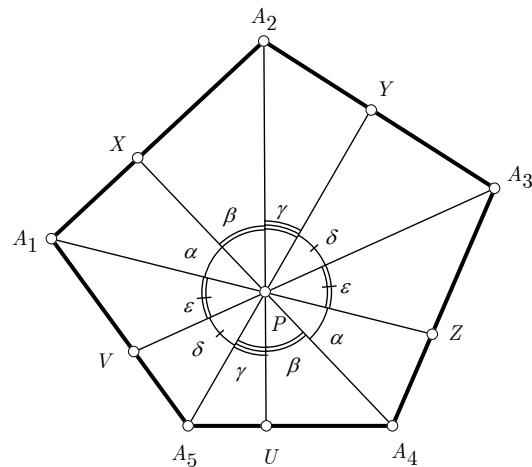
*Proof.*



In triangles  $\triangle PAB$  and  $\triangle PAC$ , the law of sines gives

$$\frac{PB}{\sin \angle PAB} = \frac{AB}{\sin \angle APB} \quad , \quad \frac{PC}{\sin \angle PAC} = \frac{AC}{\sin(180^\circ - \angle APB)} = \frac{AC}{\sin \angle APB}$$

Dividing the above relations we get the desired result. ■



Coming back to the problem, let us denote  $\angle A_1PX = \angle A_4PZ = \alpha$ ,  $\angle XPA_2 = \angle UPA_4 = \beta$ ,  $\angle A_2PY = \angle A_5PU = \gamma$ ,  $\angle YPA_3 = \angle VPA_5 = \delta$ ,  $\angle A_3PZ = \angle A_1PV = \epsilon$ , as shown in figure.

Applying the above LEMMA to the triangles  $\triangle A_1PA_2$ ,  $\triangle A_2PA_3$ ,  $\triangle A_3PA_4$ ,  $\triangle A_4PA_5$ ,  $\triangle A_5PA_1$  we get

$$\begin{aligned} & \frac{A_1X}{A_2X} \cdot \frac{A_2Y}{A_3Y} \cdot \frac{A_3Z}{A_4Z} \cdot \frac{A_4U}{A_5U} \cdot \frac{A_5V}{A_1V} = \\ & = \frac{\sin \alpha}{\sin \beta} \cdot \frac{\sin \gamma}{\sin \delta} \cdot \frac{\sin \alpha}{\sin \beta} \cdot \frac{\sin \epsilon}{\sin \alpha} \cdot \frac{\sin \beta}{\sin \gamma} \cdot \frac{\sin \delta}{\sin \epsilon} = 1 \end{aligned}$$

which ends the proof.  $\square$