Problema J147. Let $a_0 = a_1 = 1$ and

$$a_{n+1} = 1 + \frac{a_1^2}{a_0} + \cdots + \frac{a_n^2}{a_{n-1}}$$

for $n \geq 1$. Find a_n in closed form.

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We have

$$a_{n+2} = 1 + \frac{a_1^2}{a_0} + \cdots + \frac{a_n^2}{a_{n-1}} + \frac{a_{n+1}^2}{a_n} = a_{n+1} + \frac{a_{n+1}^2}{a_n}$$

Therefore, since $a_n \neq 0$ for each $n \geq 1$ (as can be proved by induction), we obtain

$$\frac{a_{n+2}}{a_{n+1}} = 1 + \frac{a_{n+1}}{a_n}$$

Thus, setting $b_n = \frac{a_{n+1}}{a_n}$, we get $b_0 = 1$ and

$$b_{n+1} = 1 + b_n, \qquad \forall n \geq 2$$

from which, clearly, it follows that $b_n = n + 1$ for each $n \geq 1$, i.e.

$$a_{n+1} = (n+1)a_n \qquad \Rightarrow \qquad a_n = n! \qquad , \qquad \forall n \geq 1$$

The proof is complete. $\square$