

Problema J152. Let $a, b, c > 0$. Prove that the following inequality holds

$$\frac{a+b}{a+b+2c} + \frac{b+c}{b+c+2a} + \frac{c+a}{c+a+2b} + \frac{2(ab+bc+ca)}{3(a^2+b^2+c^2)} \leq \frac{13}{6}$$

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Rewrite the inequality to **SOS**¹ form, as follows

$$\begin{aligned} \sum_{\text{cyclic}} \left(\frac{1}{2} - \frac{a+b}{a+b+2c} \right) + \frac{2}{3} \left(1 - \frac{ab+bc+ca}{a^2+b^2+c^2} \right) &\geq 0 && \Leftrightarrow \\ \sum_{\text{cyclic}} \frac{2c-a-b}{2(a+b+2c)} + \frac{1}{3(a^2+b^2+c^2)} \sum_{\text{cyclic}} (a-b)^2 &\geq 0 && \Leftrightarrow \\ \sum_{\text{cyclic}} \frac{c-a}{2(a+b+2c)} + \sum_{\text{cyclic}} \frac{c-b}{2(a+b+2c)} + \frac{1}{3(a^2+b^2+c^2)} \sum_{\text{cyclic}} (a-b)^2 &\geq 0 && \Leftrightarrow \\ \sum_{\text{cyclic}} \frac{a-b}{2(b+c+2a)} + \sum_{\text{cyclic}} \frac{b-a}{2(c+a+2b)} + \frac{1}{3(a^2+b^2+c^2)} \sum_{\text{cyclic}} (a-b)^2 &\geq 0 && \Leftrightarrow \\ \sum_{\text{cyclic}} \left[\frac{1}{3(a^2+b^2+c^2)} + \frac{1}{2(a+c+2b)(b-a)} + \frac{1}{2(b+c+2a)(a-b)} \right] (a-b)^2 &\geq 0 && \Leftrightarrow \\ \sum_{\text{cyclic}} \frac{a^2+10ab+b^2+6ac+6bc-c^2}{6(2a+b+c)(a+2b+c)(a^2+b^2+c^2)} (a-b)^2 &\geq 0 && \Leftrightarrow \\ S_a(b-c)^2 + S_b(c-a)^2 + S_c(a-b)^2 &\geq 0 \end{aligned}$$

in which

$$S_c = \frac{a^2+10ab+b^2+6ac+6bc-c^2}{6(2a+b+c)(a+2b+c)(a^2+b^2+c^2)}$$

and S_a, S_b are defined cyclically.

We have

$$a^2S_b + b^2S_a = \frac{f(a,b,c)}{g(a,b,c)}$$

where

$$\begin{aligned} f(a,b,c) = a^5 + 8a^4b + 9a^3b^2 + 9a^2b^3 + 8ab^4 + b^5 + 11a^4c + 32a^3bc + 22a^2b^2c + \\ + 32ab^3c + 11b^4c + 11a^3c^2 + 8a^2bc^2 + 8ab^2c^2 + 11b^3c^2 + a^2c^3 + b^2c^3 \end{aligned}$$

and

$$g(a,b,c) = 6(2a+b+c)(a+2b+c)(a+b+2c)(a^2+b^2+c^2)$$

¹The word **SOS** stands for Sum Of Squares

Therefore $a^2S_b + b^2S_a \geq 0$ for each $a, b, c > 0$ and the conclusion follows. In fact, due to symmetry between variables, we may assume without loss of generality that $a \geq b \geq c$, so

$$S_b \geq 0, \quad S_c \geq 0, \quad \frac{a-c}{b-c} \geq \frac{a}{b}$$

from which we get

$$\begin{aligned} \sum_{\text{cyclic}} S_a(b-c)^2 &\geq S_a(b-c)^2 + S_b(c-a)^2 \geq \\ &\geq S_a(b-c)^2 + \frac{a^2(b-c)^2S_b}{b^2} = \\ &= \frac{(b-c)^2(a^2S_b + b^2S_a)}{b^2} \geq 0 \end{aligned}$$

Equality holds for $a = b = c$. □