Problema J153. Find all integers n such that $n^2 + 2010n$ is a perfect square.

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We have to find all the solutions of diophantine equation

$$n^2 + 2010n = m^2, \qquad n \in \mathbb{Z}, m \in \mathbb{N}$$

which rewrites in the following equivalent form

$$\begin{aligned} n^2 + 2010n + 1005^2 = m^2 + 1005^2 \qquad &\Leftrightarrow \\ (n+1005)^2 - m^2 = 1005^2 \qquad &\Leftrightarrow \\ (n+m+1005)(n-m+1005) = 1005^2 & \end{aligned} \tag{1}$$

We can obtain the solutions of (1) by solving the following systems

$$\begin{cases} n+m = d - 1005 \\ n-m = \frac{1005^2}{d} - 1005 \end{cases}$$

where d is a divisor of $2005^2 = 3^2 \cdot 5^2 \cdot 67^2$. Hence we get

$$n = \frac{d + \frac{1005^2}{d}}{2} - 1005 = \frac{d^2 - 2010d + 1005^2}{2d} \tag{2}$$

The divisors of 2005^2 are

$$\begin{aligned} &\pm 1, \pm 3, \pm 5, \pm 9, \pm 15, \pm 25, \pm 45, \pm 67, \pm 75, \pm 201, \pm 225, \pm 335, \pm 603, \pm 1005, \\ &\pm 1675, \pm 3015, \pm 4489, \pm 5025, \pm 13467, \pm 15075, \pm 22445, \pm 40401, \pm 67335, \\ &\pm 112225, \pm 202005, \pm 336675, \pm 1010025 \end{aligned}$$

and plugging them into (2) we obtain the integers n that we are looking for:

$$\begin{aligned} &-506018, -169344, -102010, -57122, -34680, -21218, -12250, -8576, \\ &-7776, -3618, -3362, -2680, -2144, -2010, 0, 134, 670, 1352, 1608, \\ &5766, 6566, 10240, 19208, 32670, 55112, 100000, 167334, 504008 \end{aligned}$$

$\square$