Problema J154. Let ABC be an acute triangle and let $MNPQ$ be a rectangle inscribed in the triangle such that $M, N \in BC$, $P \in AC$, $Q \in AB$. Prove that

$$[MNPQ] \leq \frac{1}{2}[ABC]$$

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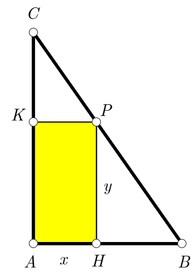
We will use the following

LEMMA. Let ABC be a triangle with $\angle A = 90°$. If $AHPK$ is a rectangle inscribed in $\triangle ABC$ such that $H \in AB$, $K \in AC$ then we have

$$[AHPK] \leq \frac{1}{2}[ABC]$$

Proof. Denote $AB = c$, $AC = b$, $AH = x$ and $PK = y$.



From similar triangles CKP and CAB we have

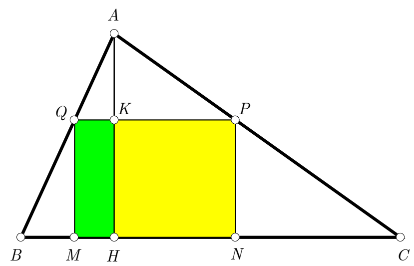
$$KP : CK = AB : CA \quad \Rightarrow \quad x : b = (a - y) : a \quad \Rightarrow \quad ax + by = ab$$

Hence

$$[AHPK] = xy = \frac{1}{ab}(ax)(by) \leq \frac{1}{ab}\left(\frac{ax + by}{2}\right)^2 = \frac{ab}{4} = \frac{1}{2}[ABC]$$

and the LEMMA is proved. ■

Coming back to the problem, draw from A the perpendicular to BC which intersects QP, BC in K, H respectively.



By applying the Lemma to the triangles ABH and AHC we get

$$[MNPQ] = [QMHK] + [HNPK] \leq \frac{1}{2}[ABH] + \frac{1}{2}[AHC] = \frac{1}{2}[ABC]$$

and the desired result is proved. $\square$