Problema J155. Find all n for which there are n consecutive integers whose sum of squares is a prime.

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Consider the sum of squares of n consecutive integers $k, k+1, \ldots, k+n-1$ and let

$$f(n,k) = \sum_{i=0}^{n-1} (k+i)^2 = \frac{1}{6} n \left(1 - 6k + 6k^2 - 3n + 6kn + 2n^2\right)$$

We want prove that the only integers n verifying the requested condition are 2,3,6. In fact

- for $n = 2$ we have $1^2 + 2^2 = 5$;
- for $n = 3$ we have $1^2 + 2^2 + 3^2 = 13$;
- for $n = 6$ we have $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 139$;

On the other hand

- for $n = 1$ we have $f(1,k) = k^2$ which is never a prime number for any value of k;
- for $n = 4$ we have $f(4,k) = 2\left(2k^2 + 10k + 15\right)$ which is never a prime number since the equation $2k^2+10k+15 = 1$ has no real solutions (because $\Delta = -12 < 0$);
- for $n = 5$ we have $f(5,k) = 5\left(k^2 + 6k + 11\right)$ which is never a prime number since the equation $k^2+6k+11 = 1$ has no real solutions (because $\Delta = -4 < 0$);
- for $n = 6t + 1$ with $t \geq 1$, we have

  $$f(6t+1,k) = (1+6t)(k^2 + 6tk + 12t^2 + t)$$

  which is never a prime number since the equation $k^2 + 6tk + 12t^2 + t = 1$ has no real solutions (because $\Delta = 4 - 4t - 12t^2 < 0$);
- for $n = 6t + 2$ with $t \geq 1$, we have

  $$f(6t+2,k) = (1+3t)\left(2k^2 + 12tk + 2k + 24t^2 + 10t + 1\right)$$

  which is never a prime number since the equation $2k^2 + (2+12t)k + 24t^2 + 10t + 1 = 1$ has no real solutions (because $\Delta = 4 - 32t - 48t^2 < 0$);
- for $n = 6t + 3$ with $t \geq 1$, we have

  $$f(6t+3,k) = (1+2t)\left(3k^2 + 18kt + 6k + 36t^2 + 27t + 5\right)$$

  which is never a prime number since the equation $3k^2 + (6+18t)k + 36t^2 + 27t + 5 = 1$ has no real solutions (because $\Delta = -12 - 108t - 108t^2 < 0$);

- for $n = 6t + 4$ with $t \geq 1$, we have

  $$f(6t+4,k) = (2+3t)\left(2k^2 + 12kt + 6k + 24t^2 + 26t + 7\right)$$

  which is never a prime number since the equation $2k^2+(6+12t)k+24t^2+26t+7 = 1$ has no real solutions (because $\Delta = -12 - 64t - 48t^2 < 0$);

- for $n = 6t + 5$ with $t \geq 1$, we have

  $$f(6t+5,k) = (5+6t)\left(k^2 + 6kt + 4k + 12t^2 + 17t + 6\right)$$

  which is never a prime number since the equation $k^2 + (4+6t)k + 12t^2 + 17t + 6 = 1$ has no real solutions (because $\Delta = -4 - 20t - 12t^2 < 0$);

- for $n = 6t$ with $t > 1$, we have

  $$f(6t,k) = t\left(6k^2 + 36kt - 6k + 72t^2 - 18t + 1\right)$$

  which is never a prime number since the equation $6k^2 + (36t-6)k + 1 - 18t + 72t^2 = 1$ has no real solutions (because $\Delta = 36 - 432t^2 < 0$).

The proof is complete. $\square$