Problema J156. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) + f(x + y)$ is a rational number for all real numbers x and all $y > 0$. Prove that $f(x)$ is a rational number for all real numbers x.

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For each real number x, consider a real number $y > 0$ and define u, v, w in the following way

$$u = f(x) + f(x + y), \quad v = f(x - y) + f(x), \quad w = f(x - y) + f(x + y)$$

Since $x = (x - y) + y$ and $x + y = (x - y) + 2y$, the numbers u, v, w are rational by hypothesis. Therefore

$$f(x) = \frac{1}{2}(u + v - w)$$

is a rational number and we are done. □