Problema J164. If x and y are positive real numbers such that $\left(x+\sqrt{x^2+1}\right)\left(y+\sqrt{y^2+1}\right)=2011$, find the minimum possible value of $x+y$.

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Suppose, more generally, that x and y are positive real numbers such that

$$\left(x+\sqrt{x^2+1}\right)\left(y+\sqrt{y^2+1}\right)=a$$

where $a \geq 1$. Thus, we have

$$\begin{aligned}
&x+\sqrt{x^2+1}=a\left(\sqrt{1+y^2}-y\right) && \Rightarrow\\
&x+ay=a\sqrt{1+y^2}-\sqrt{1+x^2} && \Rightarrow\\
&2a\sqrt{(1+x^2)(1+y^2)}=1+a^2-2axy && \Rightarrow\\
&4a^2\left(x^2+y^2\right)+4a\left(1+a^2\right)xy=\left(a^2-1\right)^2 && \Rightarrow\\
&4a^2\left(x^2+y^2+2xy\right)+4a\left(1+a^2-2a\right)xy=\left(a^2-1\right)^2 && \Rightarrow\\
&4a^2\left(x+y\right)^2+4a\left(a-1\right)^2xy=\left(a^2-1\right)^2 && \Rightarrow
\end{aligned}$$

Now, since $xy \leq \frac{(x+y)^2}{4}$ (by AM-GM inequality), we get

$$\begin{aligned}
&4a^2\left(x+y\right)^2+a\left(a-1\right)^2(x+y)^2 \geq \left(a^2-1\right)^2 && \Rightarrow\\
&(x+y)^2 \geq \frac{(a^2-1)^2}{a(a+1)^2} && \Rightarrow\\
&x+y \geq \frac{a-1}{\sqrt{a}}
\end{aligned}$$

If $a=2011$ we obtain that the minimum possible value of $x+y$ is $\frac{2010}{\sqrt{2011}}$, which is assumed if $x=y=\frac{1005}{\sqrt{2011}}$. $\square$